CRITICAL ISSUES WITH QUNATIFICATION OF DICRETIZATION UNCERTAINTY IN CFD

Ismail B. Celik, Ph.D.

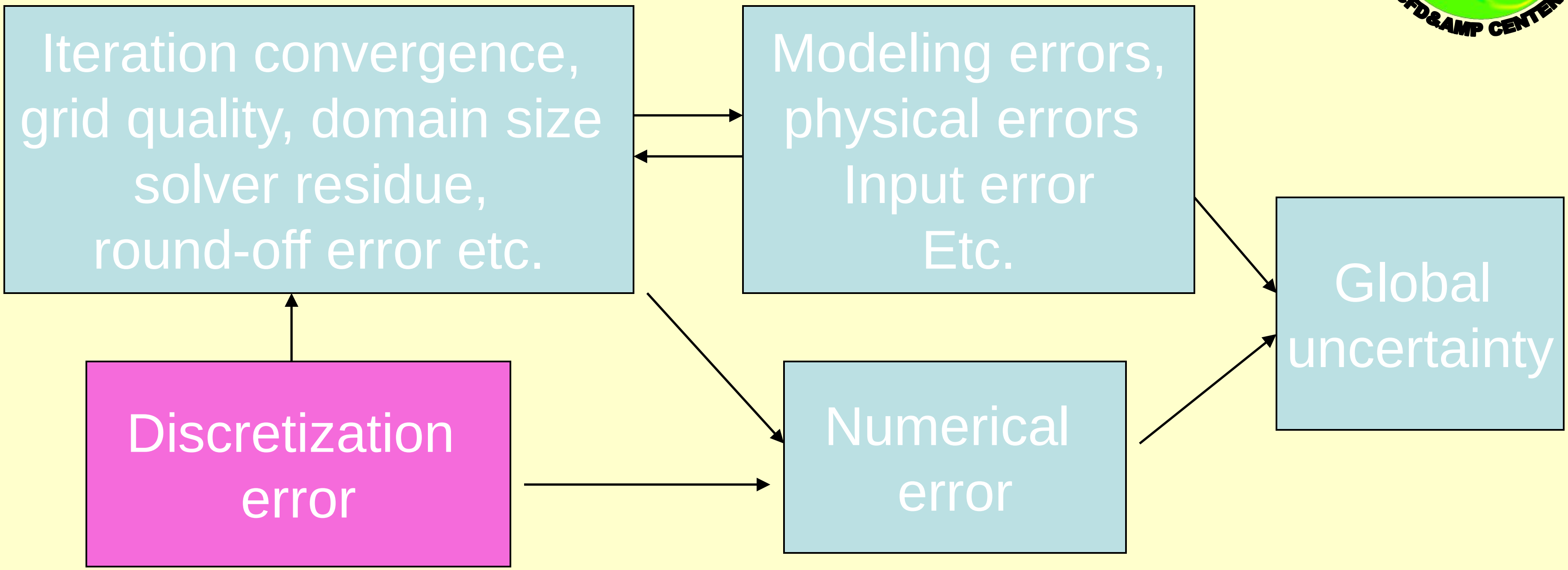
Mechanical and Aerospace Engineering Department

West Virginia University

Presented at: NETL, ***2011 Workshop on Multiphase Flow Science Airport Marriott Station Square, Pittsburg, PA***

August 16-18, 2011

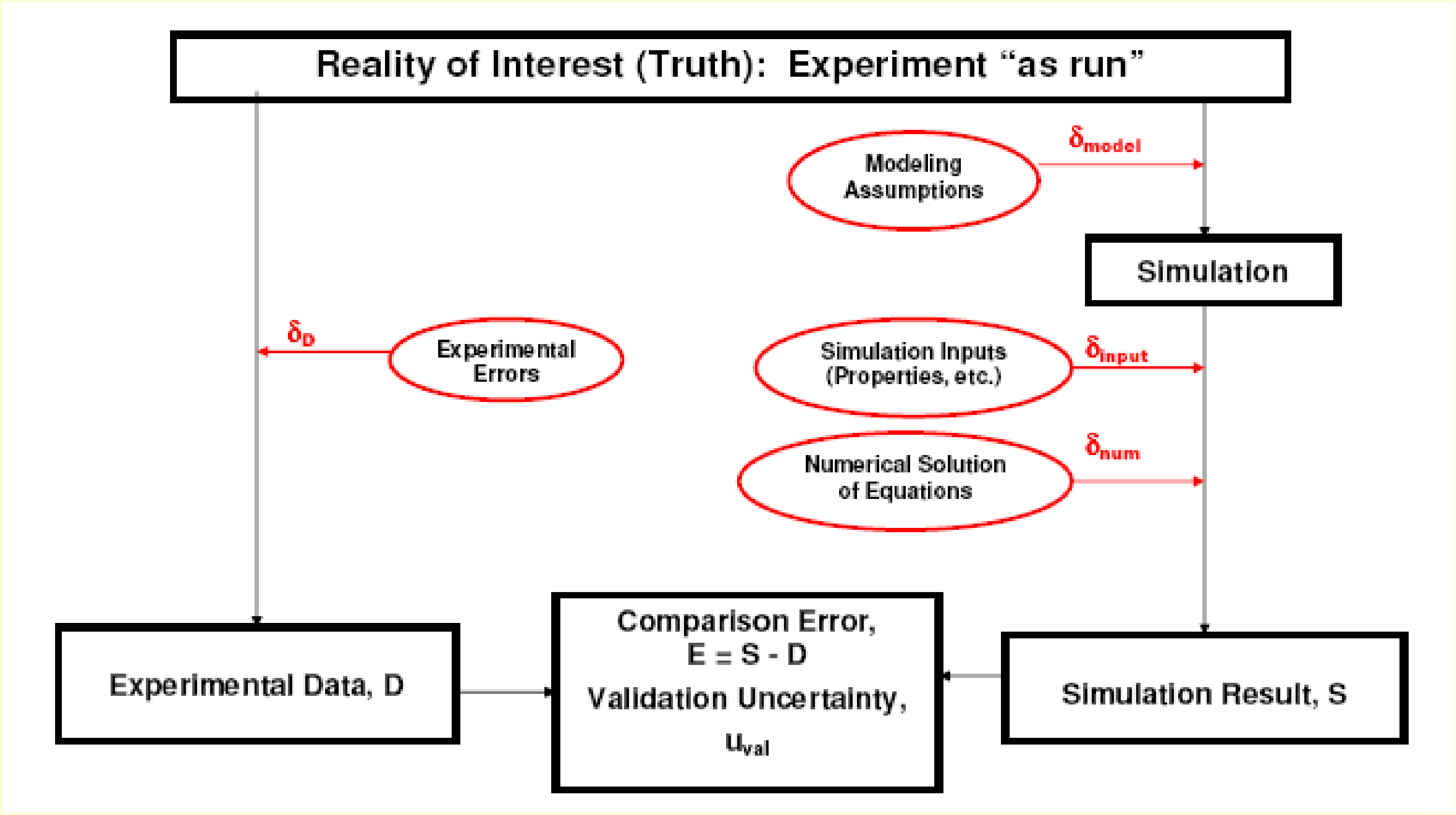
Errors and Uncertainty!



Common methods of quantifying discretization error:

- Zhu-Zienkiewicz (ZZ) and energy norm methods
- Richardson extrapolation (RE)
- Error transport method (ETE)
- Hybrid ETE and Residual Methods

V&V Overview*



$$E = \delta_{\text{model}} + \delta_{\text{input}} + \delta_{\text{num}} - \delta_{\text{D}} \qquad u_{\text{val}} = u_{\text{D}}^2 + u_{\text{num}}^2 + u_{\text{input}}^2 \;{}^{\frac{1}{2}}$$

****After Coleman (see e.g. ASME V&V 20, 2010)***

On the question of determinism

- " … the randomness of quantum mechanics is like a coin toss*. It looks random, but it's not really random."

  Carsten van de Bruck

  – from Musser , G. (2004) 'Was Einstein Right?' Scientific American September issue, pp. 88-91

  – * A*ll coins tossed from a skyscraper with different initial velocities will reach the same terminal velocity due to friction loss (i.e. information loss)*

Error Analysis: Deterministic Methods

Goal: Assessment of all types of numerical errors and modeling errors with repeatable (deterministic) calculations.

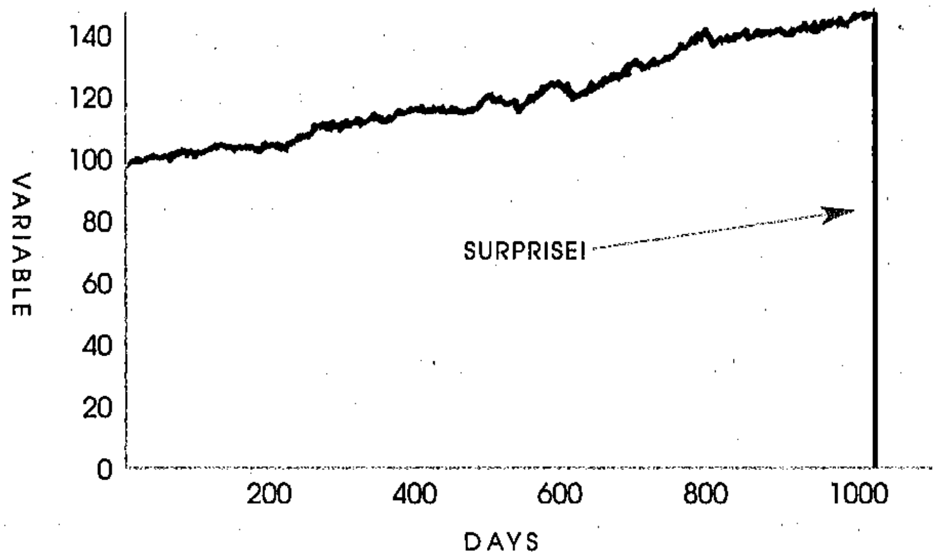
<ins>Calculation Verification:</ins> A calculation is what it is supposed to be in the context of numerical analysis, i.e. the equations (PDE's) are solved right! (After P. Roache)
In practice: Assess grid convergence

<ins>Validation:</ins> Assessment of modeling errors in conjunction with verification: Compare with Experiments, DNS, Observations, Perceptions

- ***Paradox: Determinism ←→ Randomness/surprise/unpredictable***

The Black Swan Phenomenon:
White swans, gray swans, and black swans

FIGURE 1: ONE THOUSAND AND ONE DAYS OF HISTORY



A turkey before and after Thanksgiving. The history of a process over a thousand days tells you nothing about what is to happen next. This naïve projection of the future from the past can be applied to anything.

Ref.: *The Black Swan, The impact of the highly improbable by N.N. Taleb, 2010, Random House*

Numerical Dissipation is Always There!

- Theoretical analysis by Ghosal *(1996, J. Comp. Phys, 125, pp. 187-206)* concludes:
  - **Finite Diff. Error = const * λ^q** ; λ is the wave number, *q = const = 0.75 and independent of the scheme, const varies with the scheme (1.03 for 2nd order CD, 0.5 for 8th order CD)*

- Choi and Moin (1994): 2nd order methods have certain advantages, and 'higher order' is not necessarily better.

- Even with higher order methods Numerical dissipation can be as large as the modeling error, and may cancel each better.

<u>**The key question**</u>: ***How do the numerical errors interact with modeling errors***?

Uncertainty of Numerical Solutions

Implicit Solution | **Explicit Solution**

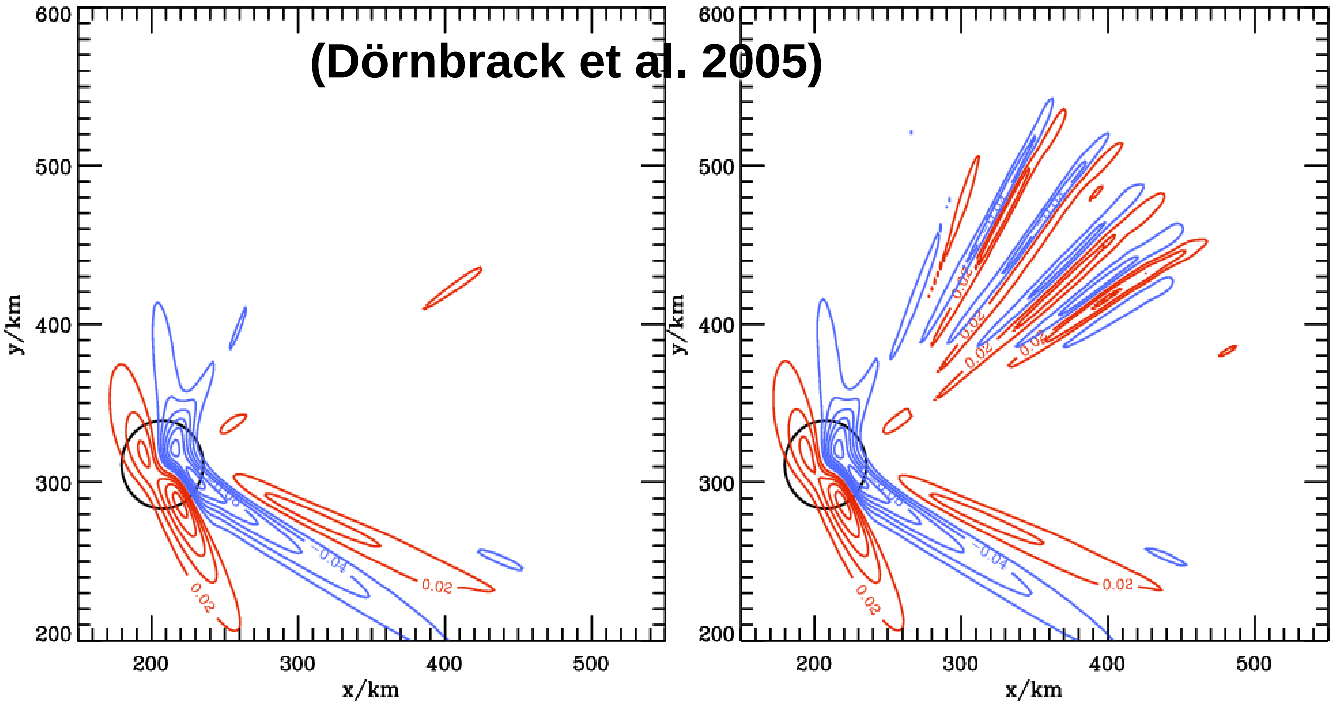
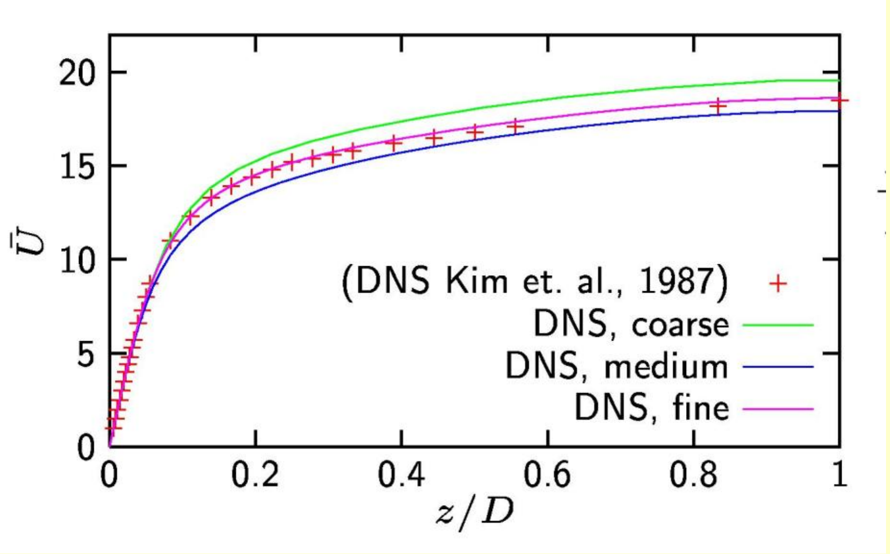
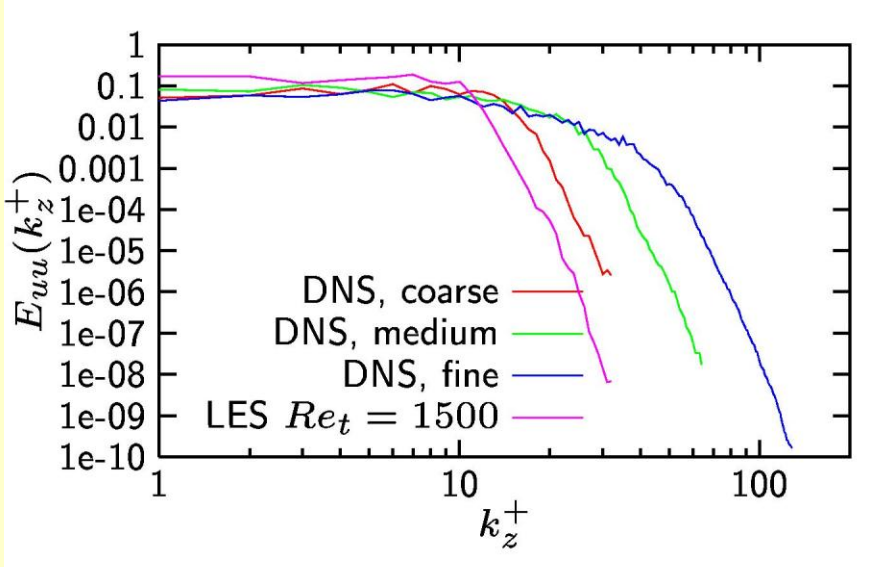


Figure 1: Vertical velocity with contour interval $\Delta w = 0.02\,\mathrm{m/s}$ (red positive, blue negative) at $z = 6\,\mathrm{km}$ altitude, after 6 h integration time for the implicit (left) and explicit simulation (right). The black line marks the 100 m elevation contour line.

Numerical dissipation & Effective Re



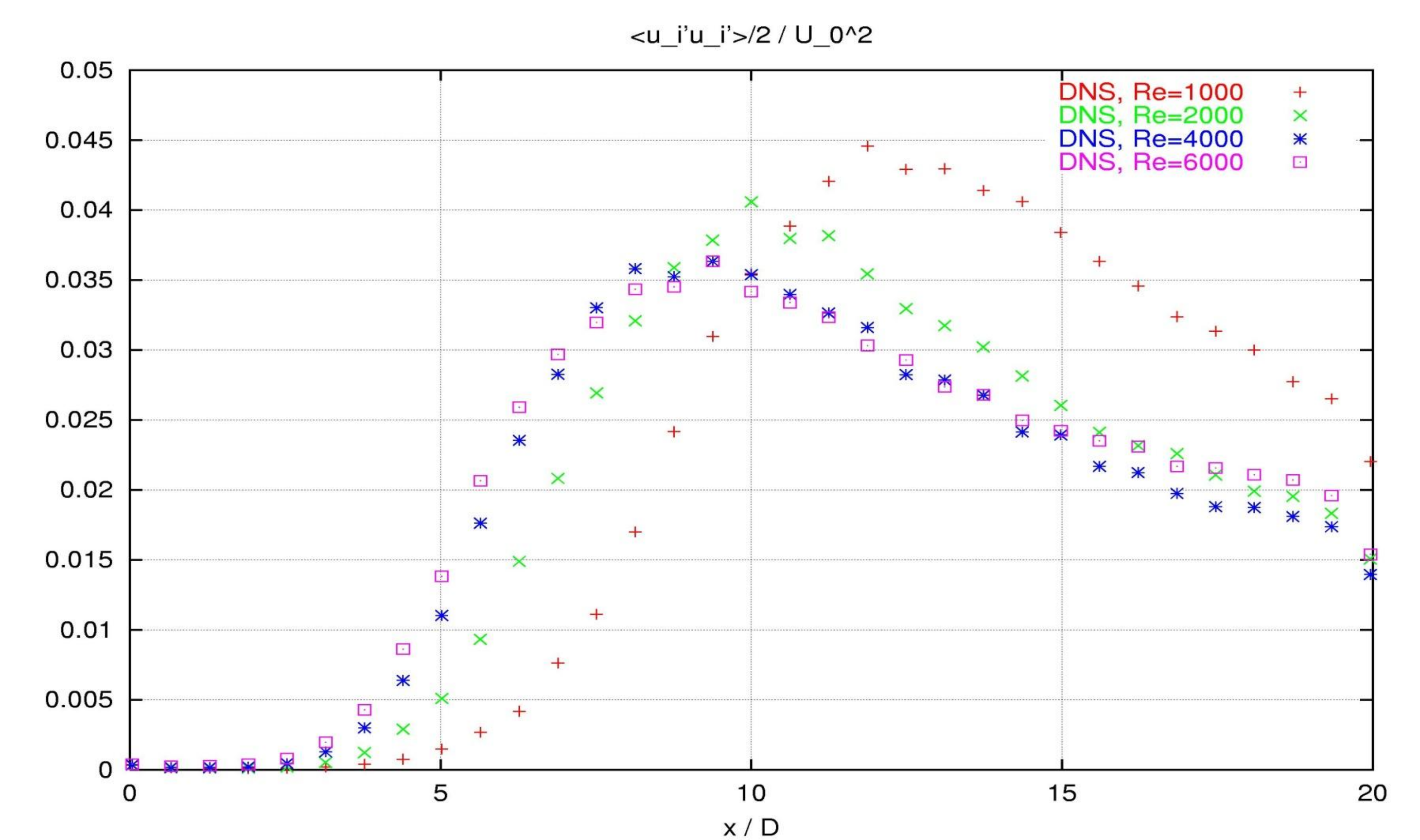
Mean velocity profile



Energy spectra

Data: ***Courtesy of Dr. Ing. Markus Klein (2005)*****; Channel flow**

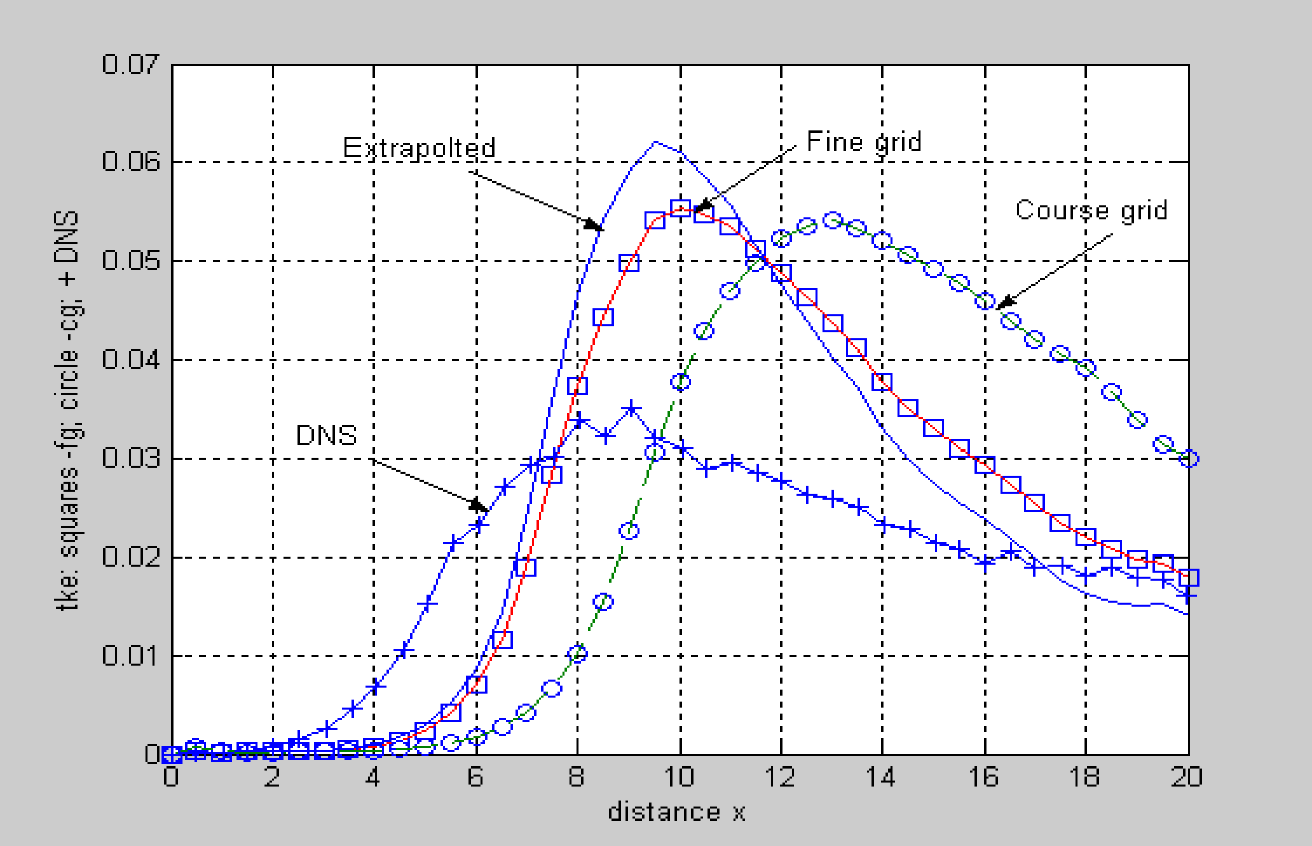
Transition in a Plane jet:



Turbulent kinetic energy profiles for a plane jet: DNS

Data: ***courtesy of Dr. Ing. Markus Klein (2005)***

Transition in a Plane jet & Eff-Viscosity

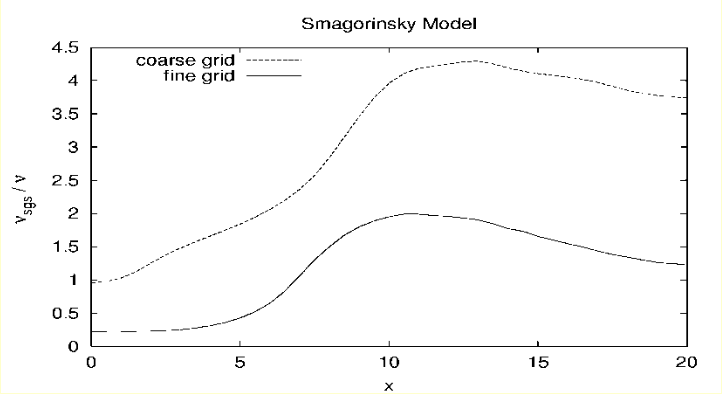


SSM

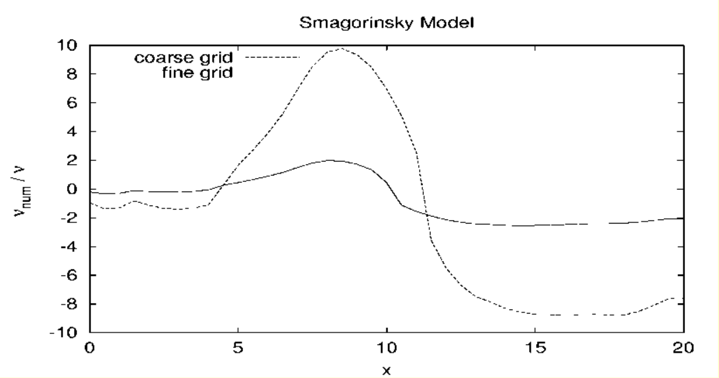
Comment: resolved tke should increase as grid is refined, or should it ?

Resolved turbulent kinetic energy profiles, (Klein et el, 2005); Smagorinsky model, Re = 4000 (based on inlet velocity =1.0 m/s and nozzle diameter = 1.0m)

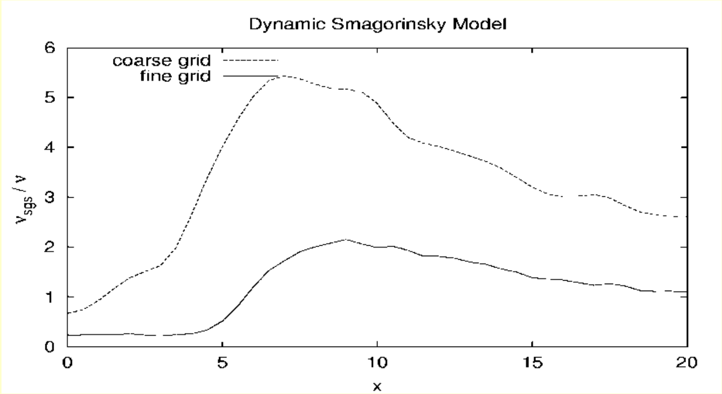
Importance of Numerical Viscosity



a

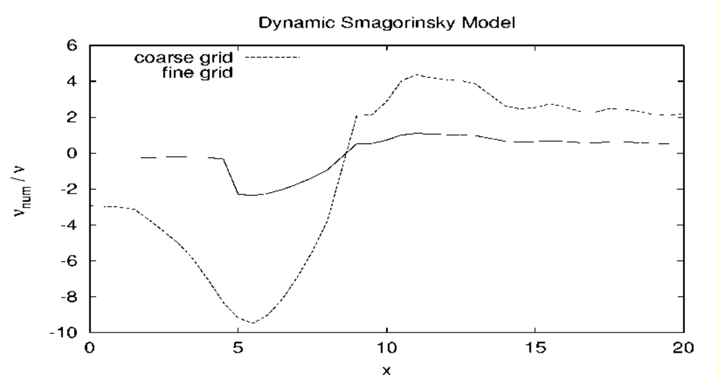


a



b

Sgs-viscosity obtained from plane jet LES data.

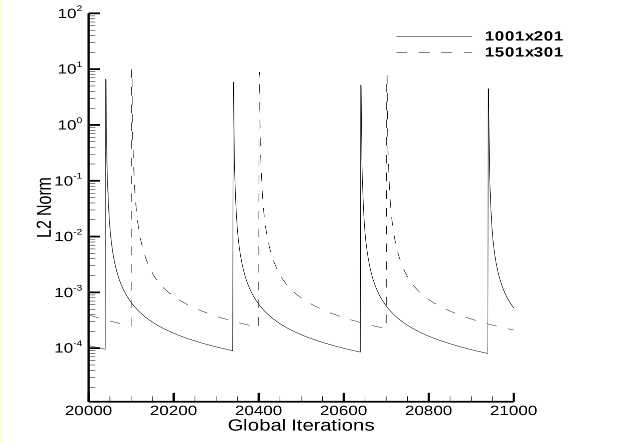


b

Estimated numerical viscosity normalized by laminar viscosity for plane jet LES data: (a) SSM, (b) DSM

Error Analysis: Iterative Convergence

Goal: Reduce normalized residual 3-4 orders of magnitude



Variation of L_2-norm with iterations (after Huebsch, 2005)

- L2_norm of approximate iteration error > L2(Residuals)
- Eigenvalue of the solution matrix is important; Approximate iteration error is given by

$$\varepsilon_{iter}^{n} \cong \frac{\left| \phi^{n+1} - \phi^{n} \right|}{\lambda_1 - 1} \qquad \lambda_1 \cong \frac{\left\| \phi^{n+1} - \phi^{n} \right\|}{\left\| \phi^{n} - \phi^{n-1} \right\|}$$

Error Analysis: Grid Convergence

Goal: Quantification of discretization errors

$$\phi_h = \phi_0 + \sum_{k=1}^{\infty} C_k \quad x_h \quad h^k \qquad\qquad C_k \quad x_h \quad = \left(\frac{\partial \phi_h}{\partial h} \right)_{h=0}$$

$$h = \quad \Delta x \Delta y \Delta z \Delta t^{*} \quad {}^{1\!/\!4} \qquad\qquad h = \quad \Delta x^2 + \Delta y^2 + \Delta z^2 + (\Delta t^{*})^2 \quad {}^{1\!/\!2}$$

$$\phi_h = \phi_{ext} + c h^p \qquad\qquad \phi_h = \phi_{ext} + c_1 h + c_2 h^2$$

$$\Delta t^{*} = u_{ch} \Delta t$$

(3 grid study is needed to determine p, c, and ϕ)

Error Analysis: Richardson Extrapolation-1

$$\phi = \phi_{ext} + ch^{p} \qquad \text{; observed order p} \qquad p = \frac{\ln\left\{\dfrac{|\phi_3 - \phi_2|}{|\phi_2 - \phi_2|}\right\}}{\ln \ r}$$

Variants: Celik et. al. (2005), Eca and Hoekstra (2004), Orozco et al (2004), Celik and Karatekin (1997),
e.g.: Restrict 0<p<5; but p=-6 means something (see next slide)

Perform at least 4-grid calculations and treat the outcome-as statistically random outcomes (Least squares, Eca et al, 2003-2004).

Richardson Extrapolation & Numerical Uncertainty: GCI Proposed by P. Roache

$$GCI = \frac{F_s}{r^p - 1}\left|f_2 - f_1\right|; \quad GCI\% = \frac{GCI}{\left|f_1\right|}$$

Table 1. Proposed implementation of the GCI for solutions on three or more systematically-refined grids using Equation (47).

$\left\lvert\frac{\hat{p} - p_f}{p_f}\right\rvert$	F_s	p
≤ 0.1	1.25	p_f
> 0.1	3.0	$\min\left(\max\left(0.1, \hat{p}\right), p_f\right)$

Using a global order works better!

Uncertainty estimation methods

- Grid Convergence Index (GCI)

$$U_\phi^f = 1.25\left|\frac{\phi_{ext} - \phi_f}{\phi_f}\right|$$

- Coefficient of variation

For least squares

$$\sigma_r = \sum_{i=1}^{n}[\phi_i - (\phi_{ext} + \alpha h_i^p)]$$

standard error of the fit

$$\sigma_{\phi/h} = \sqrt{\sigma_r/(n-3)}$$

$$CV = \left|\frac{\sigma_{\phi/h}}{\phi_{ext}}\right|$$

For the other methods using triplets

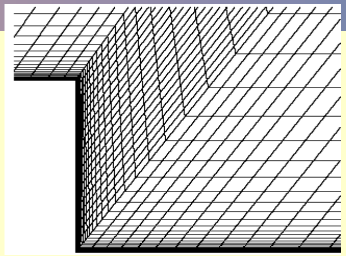
$$\mu = \sum_{i=1}^{n}\phi_{ext,i}/n$$

$$\sigma = \sqrt{\sum_{i=1}^{n}(\phi_{ext,i} - \mu)^2/(n-1)}$$

$$CV = \left|\frac{\sigma}{\mu}\right|$$

$$ERE_{CV} = ERE + CV$$

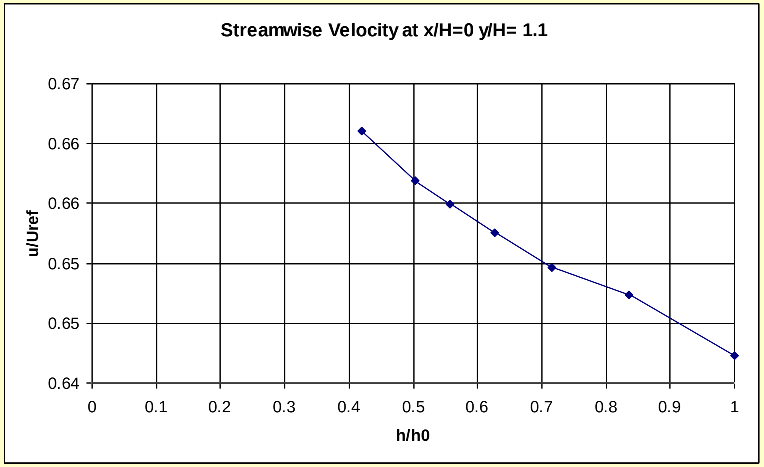
Example: Backward Step Flow



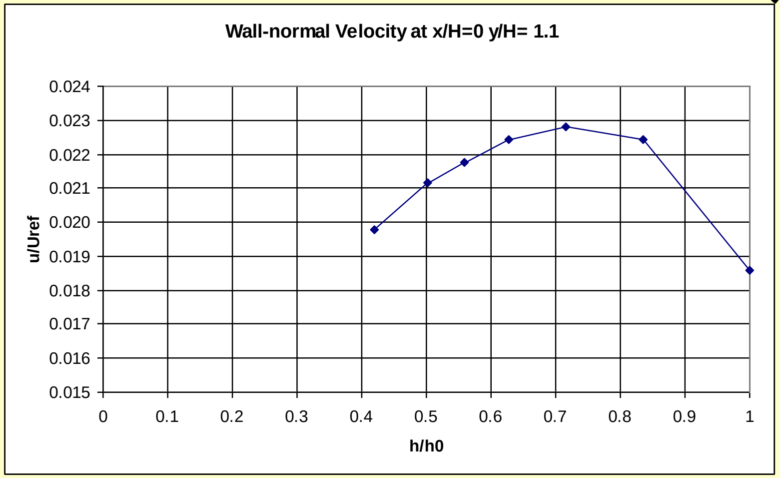
grids	ratio		grids	Ratio		Grids	Ratio		grids	ratio
101*101			101*101			141*141			101*101	
121*121	1.20		141*141	1.40		181*181	1.29		141*141	1.40
141*141	1.17		201*201	1.43		241*241	1.33		181*181	1.29
161*161	1.14								241*241	1.33
181*181	1.13									
201*201	1.11									
241*241	1.20									

•The four sets of grids used to calculate the extrapolation with least square method are 101-141-181-241

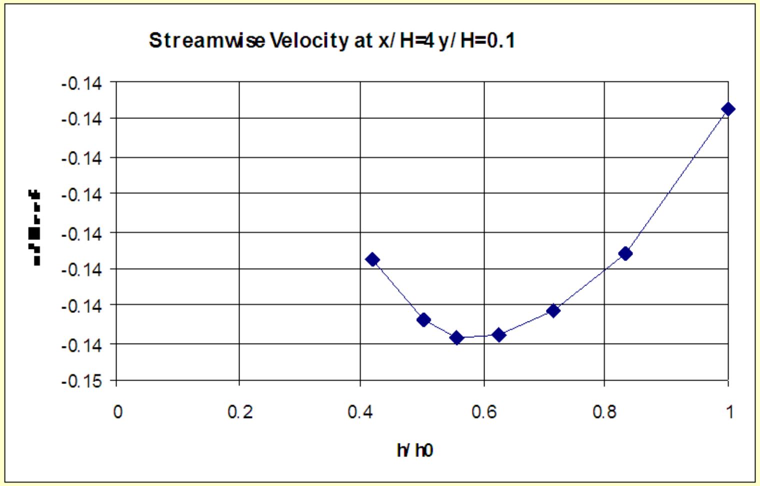
Convergence patterns --monotonic and oscillatory



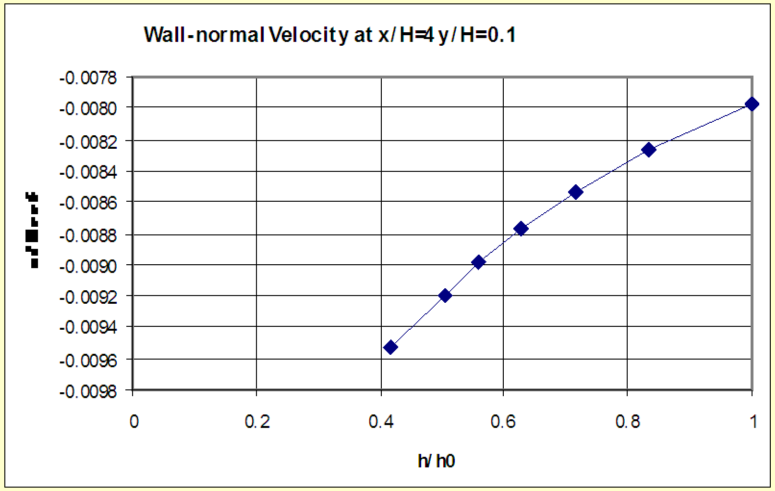
Streamwise vel. at (0,1.1)



Wall-normal vel. at (0,1.1)



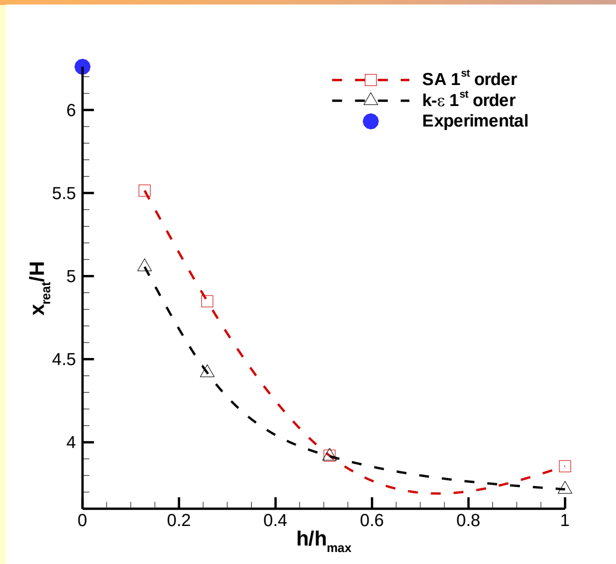
Streamwise vel. at (4,0.1)

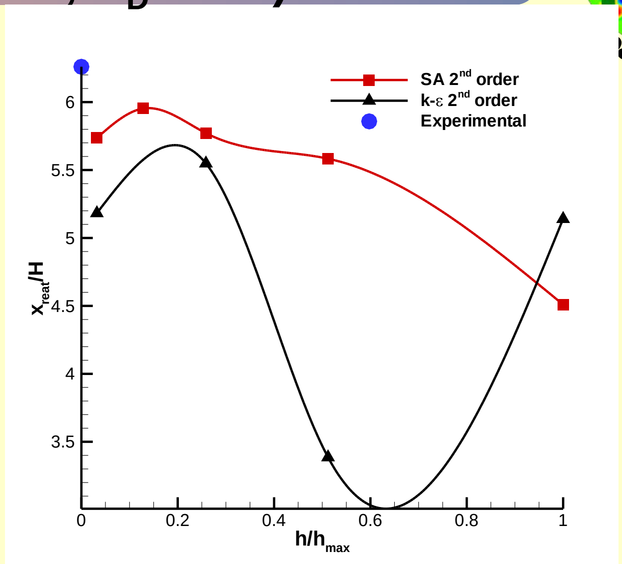


Wall-normal vel. at (4,0.1)

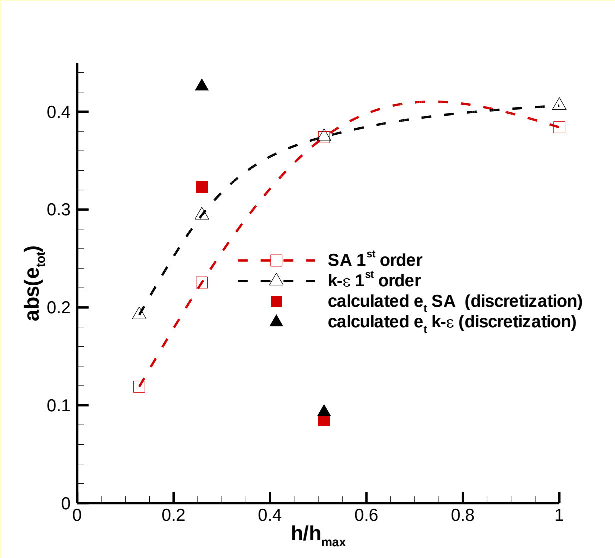
Flow Over a Backward Facing Step

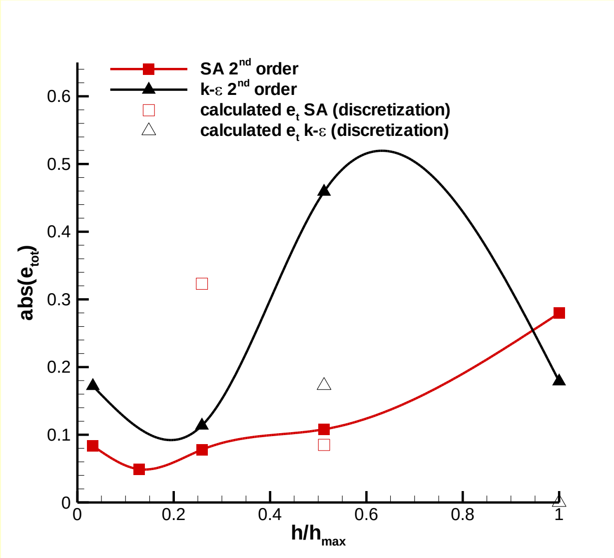
Reattachment Point (D=6.26, U_D=10%)





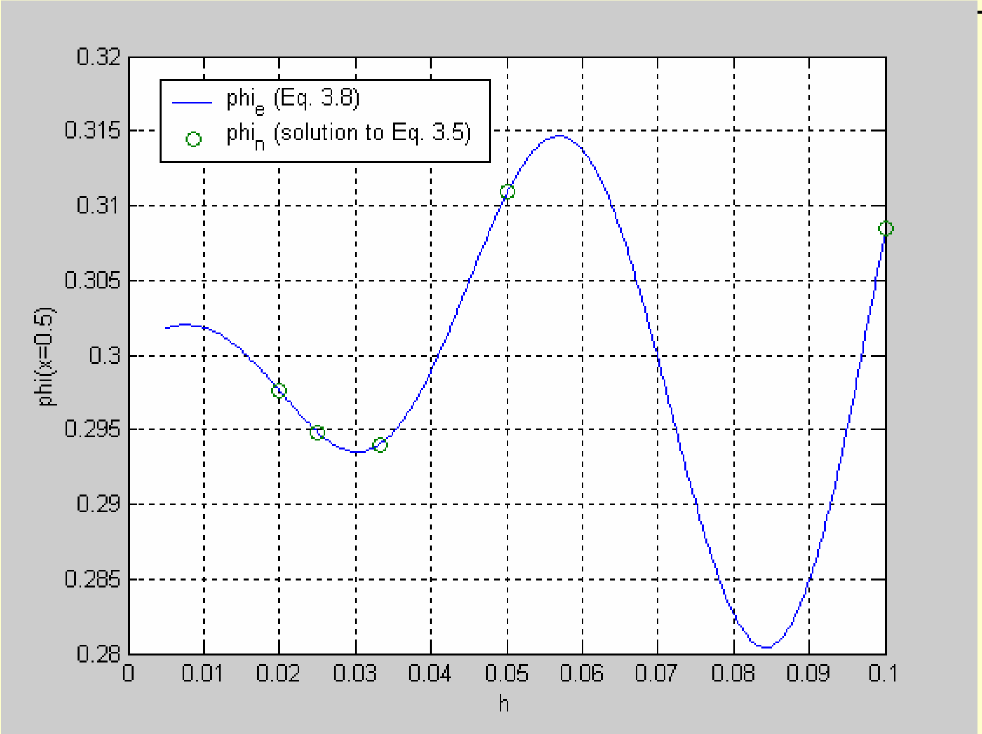
Total Error in Estimation of Reattachment Point





Oscillatory Convergence: Manufactured FDE (Celik etal, 2005)

$$u\phi_x = \phi_{xx} - \lambda\phi \qquad \text{with } \phi(0)=0 \quad \phi(1)=1$$



$$-a_i\tilde{\phi}_{i-1} + b_i\tilde{\phi}_i - c_i\tilde{\phi}_{i+1} = 0$$

$$b_i = a_i + c_i + \lambda \qquad (1)$$

Assume $a_i = c_i + \dfrac{u_i}{h}$ (2)

$$E_i \equiv \tilde{\phi}_i - \phi_i = g_i f$$

$$f = h^p \cos(kh)$$

$$g_i = \beta(i-1)(nx-i)$$

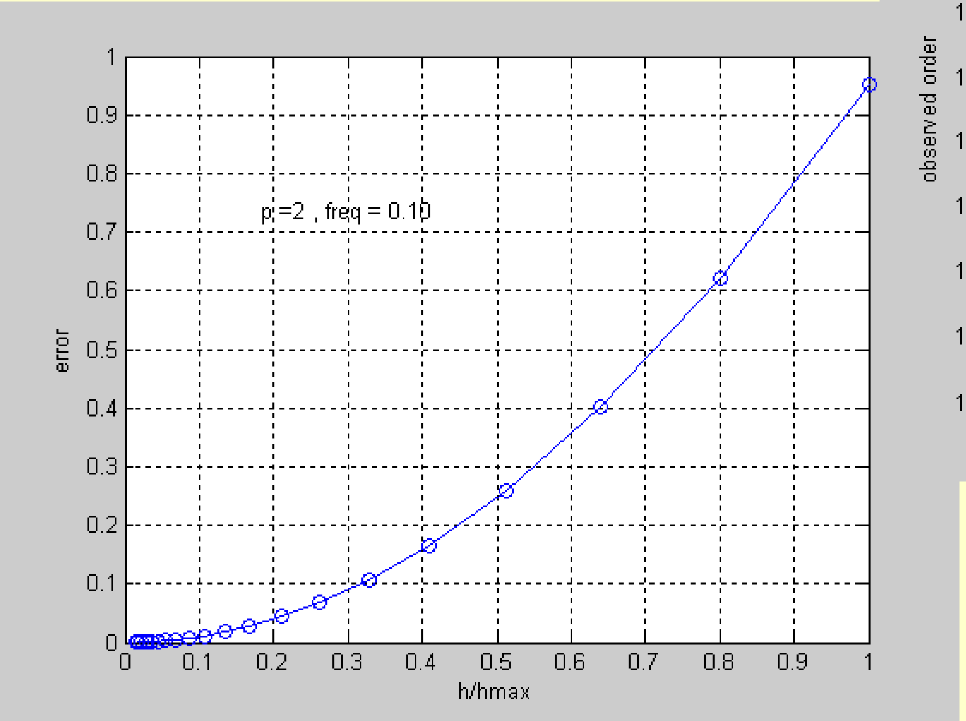
$$-a_i(\phi_{i-1} + g_{i-1}f) + b_i(\phi_i + g_i f) - c_i(\phi_{i+1} + g_{i+1}f) = 0 \qquad (3)$$

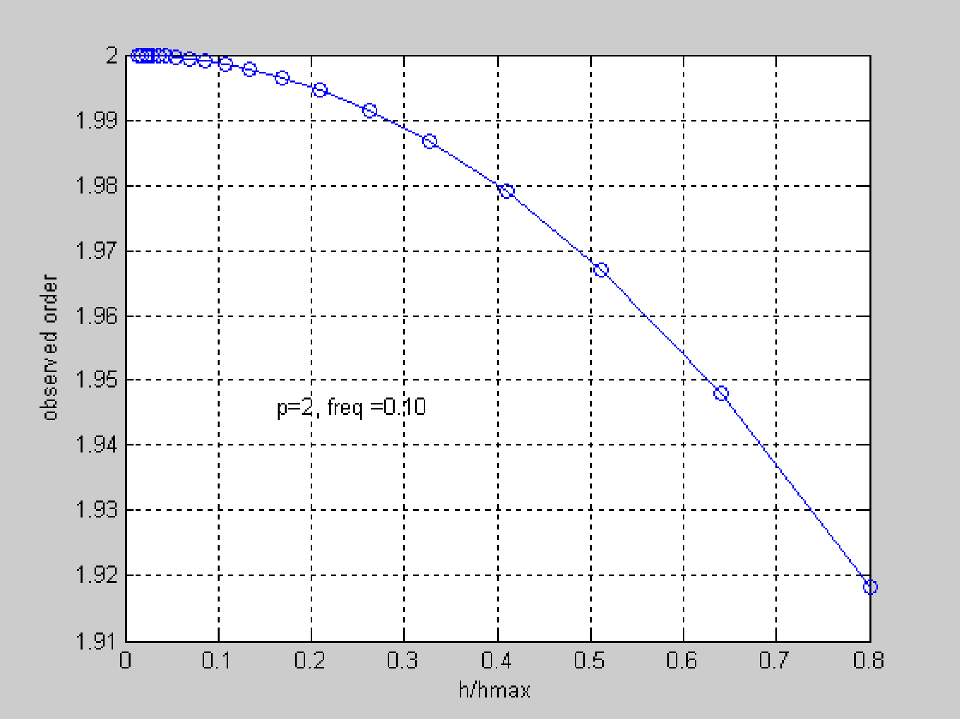
a_i, b_i and c_i can be solved by combining (1-3)

Error Analysis: Richardson Extrapolation-4

Oscillatory convergence examples

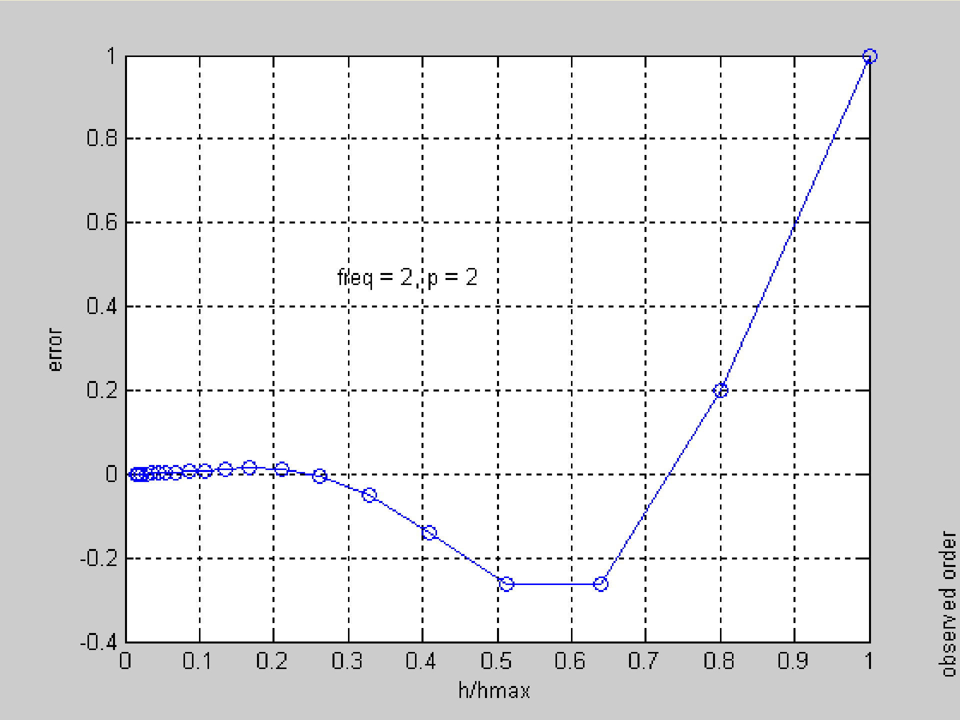
$$\phi_h = \phi_0 + gh^p \cos\ 2\pi fh$$

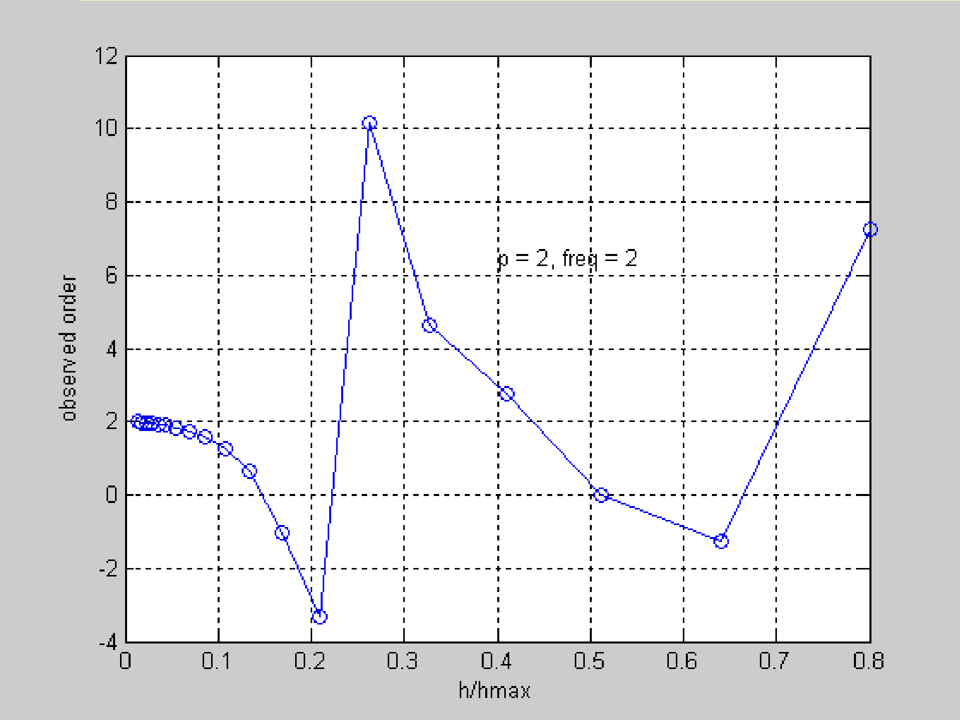




Error Analysis: Richardson Extrapolation-5

Oscillatory convergence examples





AES: An Alternative Error estimation method (Celik et al, 2008):

Present method assumes that the true error E_t is proportional to the approximate error, E_a

$$E_t^h = cE_a^h$$

For a three grid (triplet) calculation:

True error: $E_t^h = \phi - \phi_h$

Approximate error: $E_a^h = \phi_h - \phi_{\alpha h}$

$\alpha_1 = h_2/h_1$

$\alpha_2 = h_3/h_2$

$h_1 < h_2 < h_3$

Local proportionality constants:

$$c_{i,j}^1 = \frac{\phi_2^{i,j} - \phi_1^{i,j}}{\phi_3^{i,j} - 2\phi_2^{i,j} + \phi_1^{i,j}}$$ Fine-medium meshes

$$c_{i,j}^2 = \frac{\phi_3^{i,j} - \phi_2^{i,j}}{\phi_3^{i,j} - 2\phi_2^{i,j} + \phi_1^{i,j}}$$ Medium-coarse meshes

Global proportionality constant: $$c = \frac{1}{2}\frac{\left\|c_{i,j}^1\right\|_\infty + \left\|c_{i,j}^2\right\|_\infty}{N}$$

$$\left\|c_{i,j}\right\|_\infty = \sum_{k=1}^{N}\left|c_{i,j}\right|$$

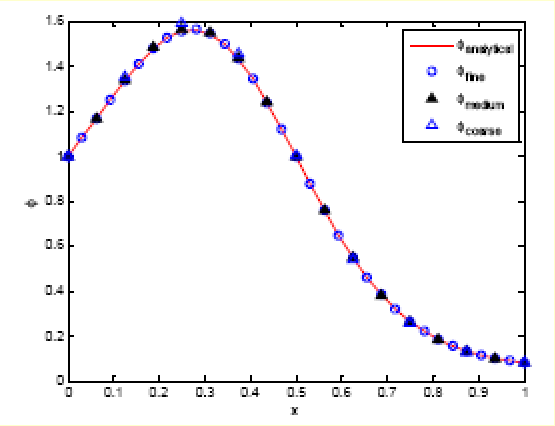
N: # of grid points

Example Application of the AES(Approximate Error Scaling Method

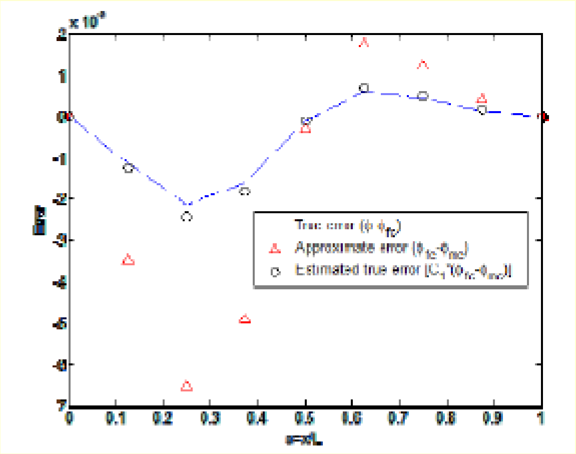
$$\frac{d}{dx}(u\phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) + S_\phi$$

$$u = \overline{u}\cos(\omega x)$$

$$\phi = \exp\left(\frac{ux}{\Gamma}\right)$$

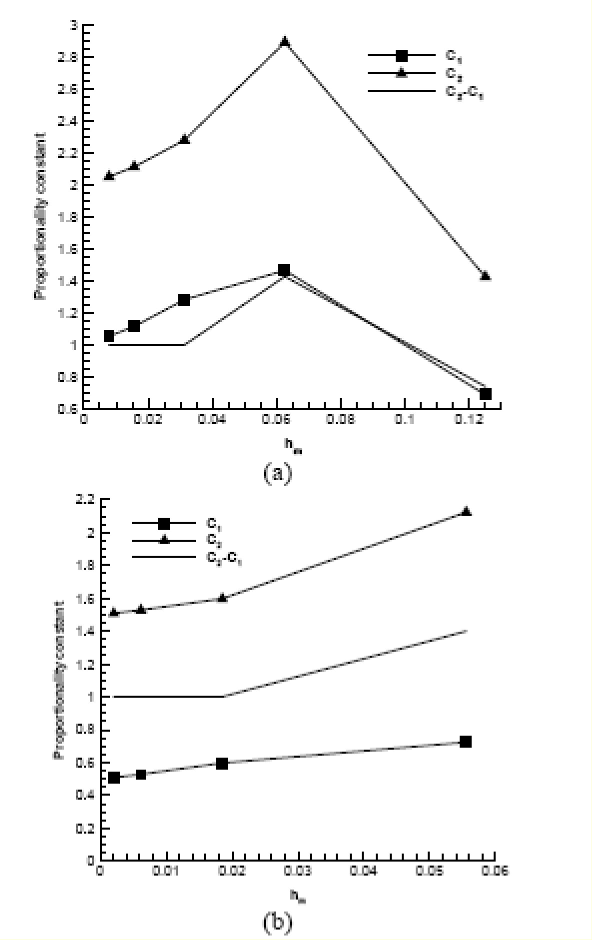


Analytical and numerical solutions of the scalar

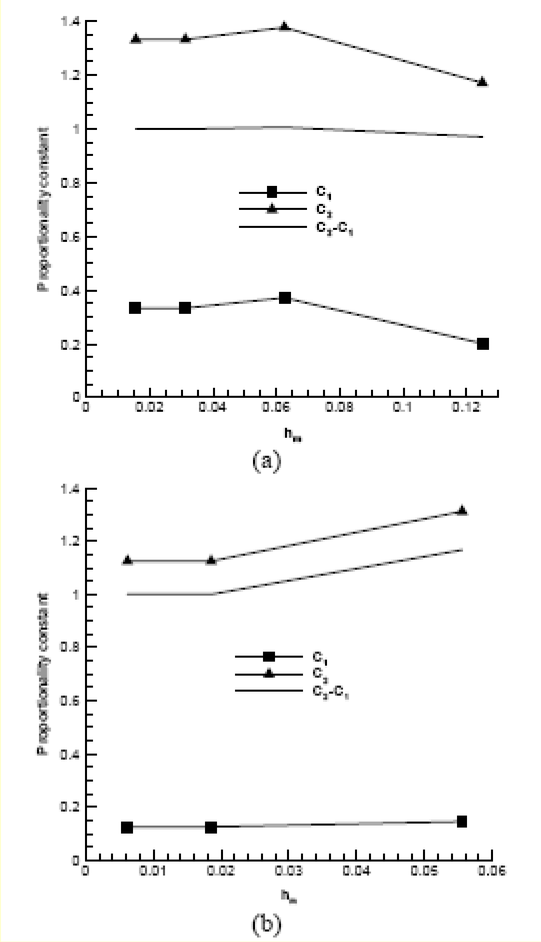


True error, approximate error and estimated true error

Approach to Asymptotic Range



Upwind scheme for convective terms with refinement factors of (a) 2 and (b) 3

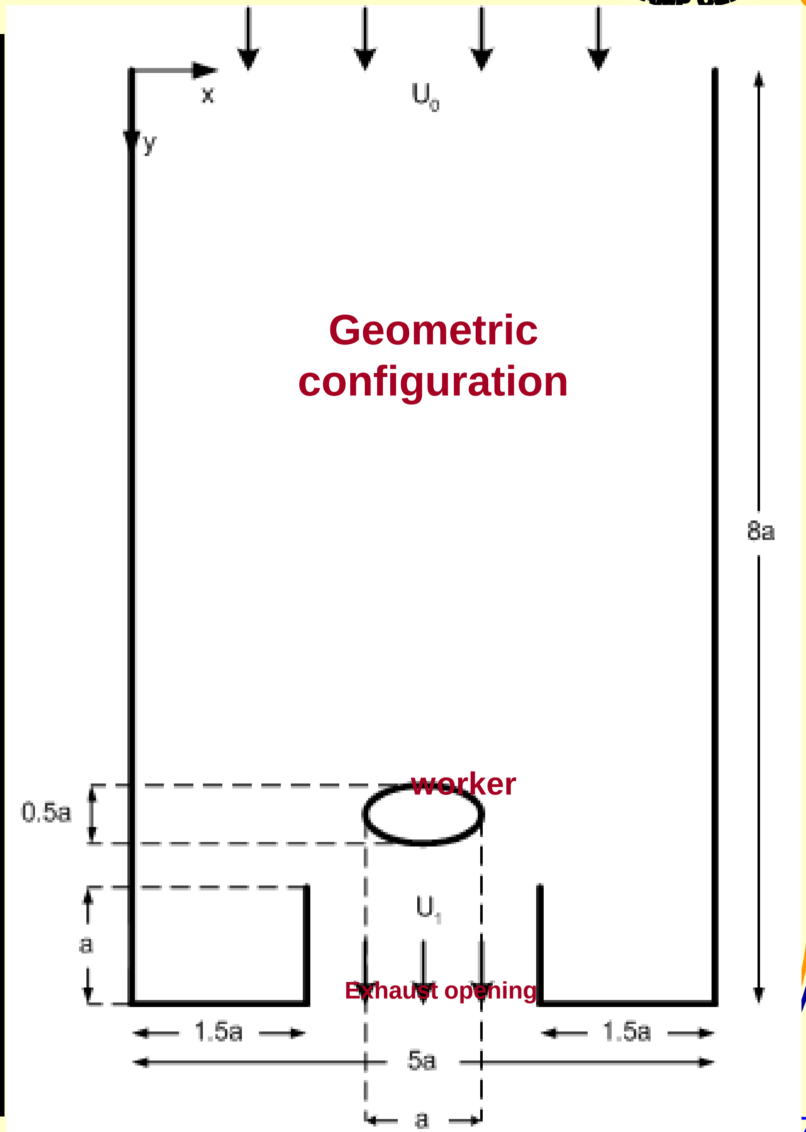


Central differencing for convective terms with refinement factors of (a) 2 and (b) 3

2D flow around an ellipse (with contraction in the downstream)

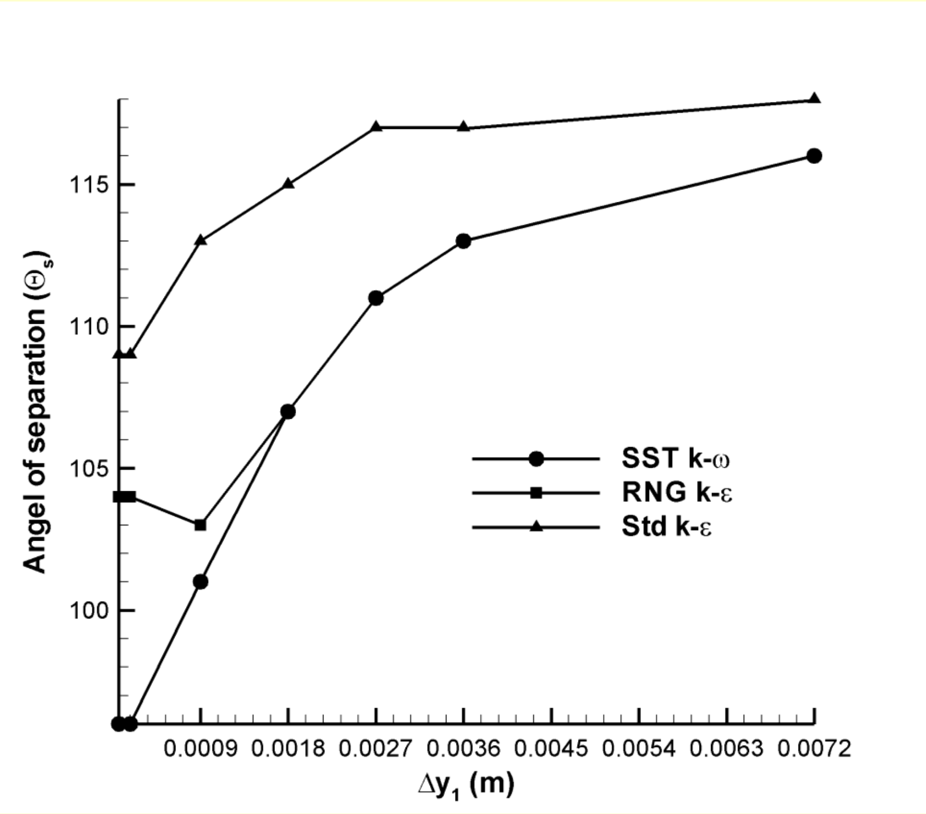
Proposed by Dunnett (1994)

Simulation Details	
Reynolds number $Re = aU_0/2\nu_{air}$	**100,000** **10,000** **1,000**
Turbulence models	**Standard k-ε** **RNG k-ε** **SST k-ω**
Inlet velocity (U_0)	**5m/s** **0.5m/s** **0.005,/s**
Time step	**0.006 s**
Total time	**40 s**
Vel.-Pres. Coupling	**SIMPLEC**
Scheme	**QUICK for conv.** **2nd order cent for diff**
Residuals	**1x10^{-4}**



2D flow around an ellipse (with contraction in the downstream)

Grid dependency of parameters relevant to flow separation



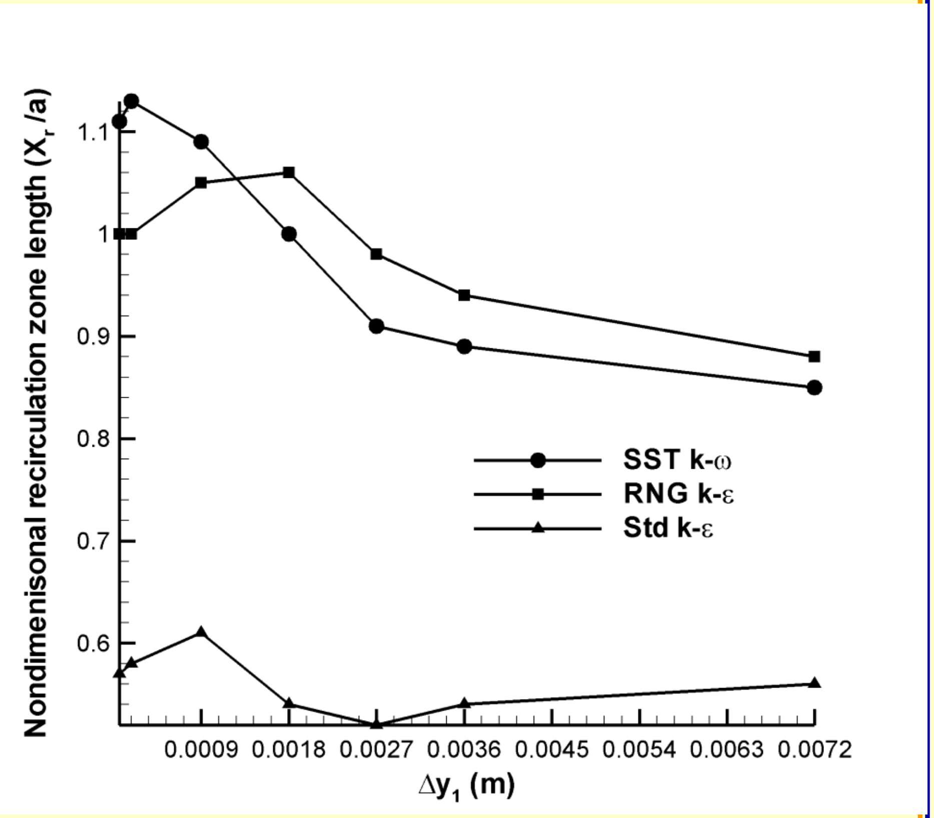
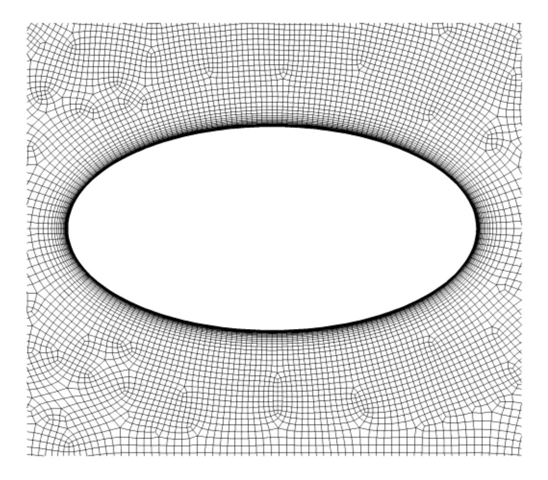


Table: Number of cells used in two-dimensional simulations

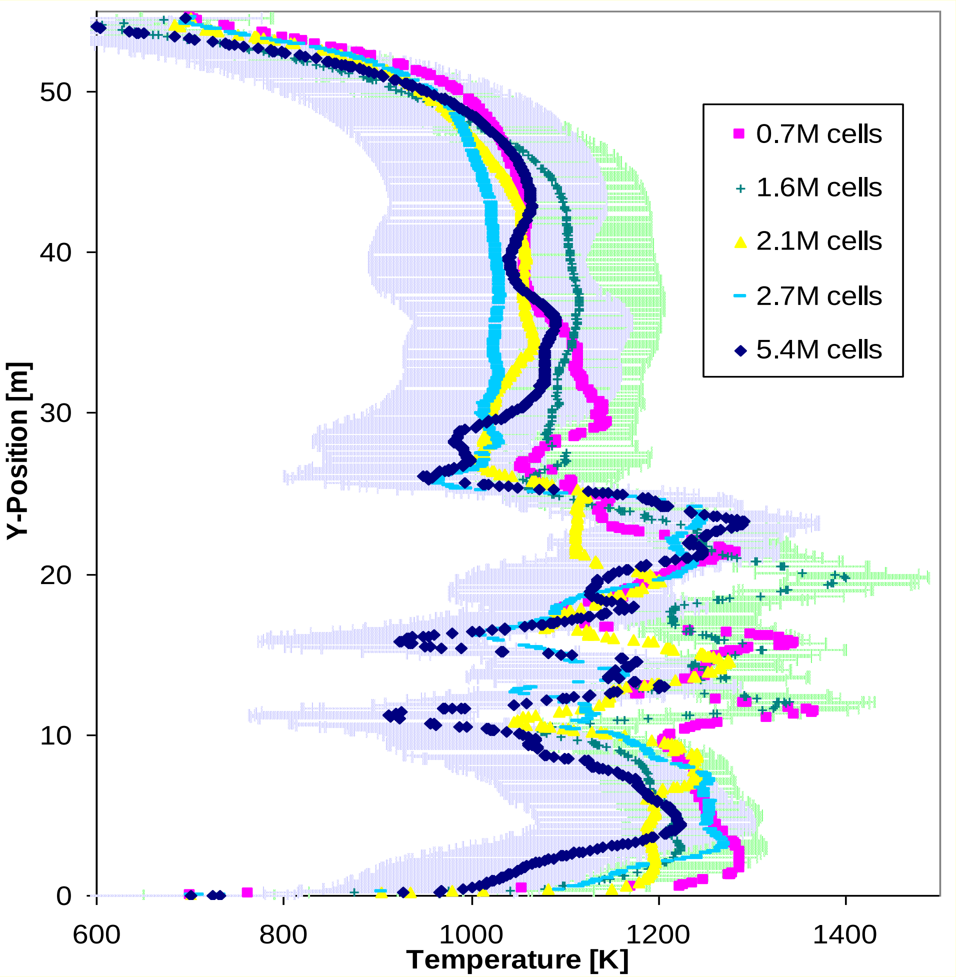
Grid	Re = 1.0x10^5	Re = 1.0x10^4	Re = 1.0x10^3
G1	9,594 (Δy_1=7.2x10^{-3}m)	51,144[2] (Δy_1=4.6x10^{-4}m)	51,144[2] (Δy_1=4.6x10^{-4}m)
G2	24,993 (Δy_1=3.6x10^{-3}m)		
G3	49,037 (Δy_1=2.7x10^{-3}m)		
G4	59,340[1] (Δy_1=1.8x10^{-3}m)		
G5	76,663[1] (Δy_1=9.0x10^{-4}m)		
G6	81,081[1] (Δy_1=1.8x10^{-4}m)		
G7	93,883[1] (Δy_1=6.0x10^{-5}m)		

[1]: Enhanced wall treatment used in k-ε model calculations

[2]: Transitional flow modifications are enabled in SST k-ω model calculations



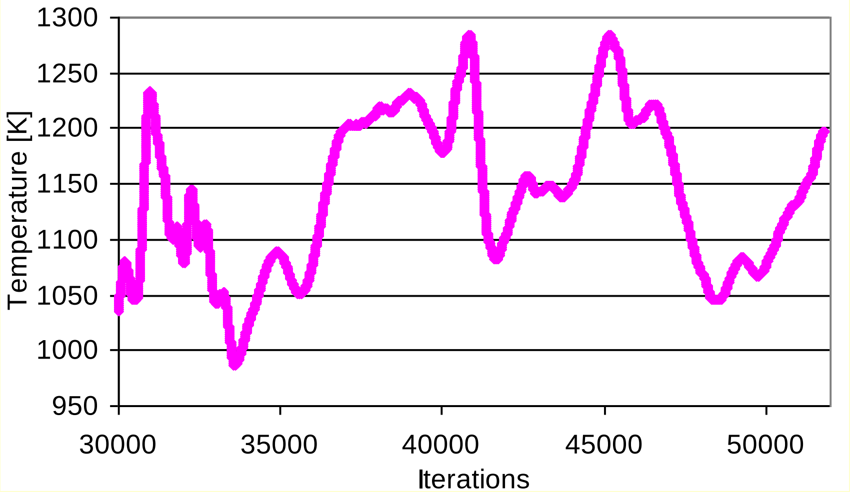
650MW Boiler Simulations



Temperature along the symmetrical line

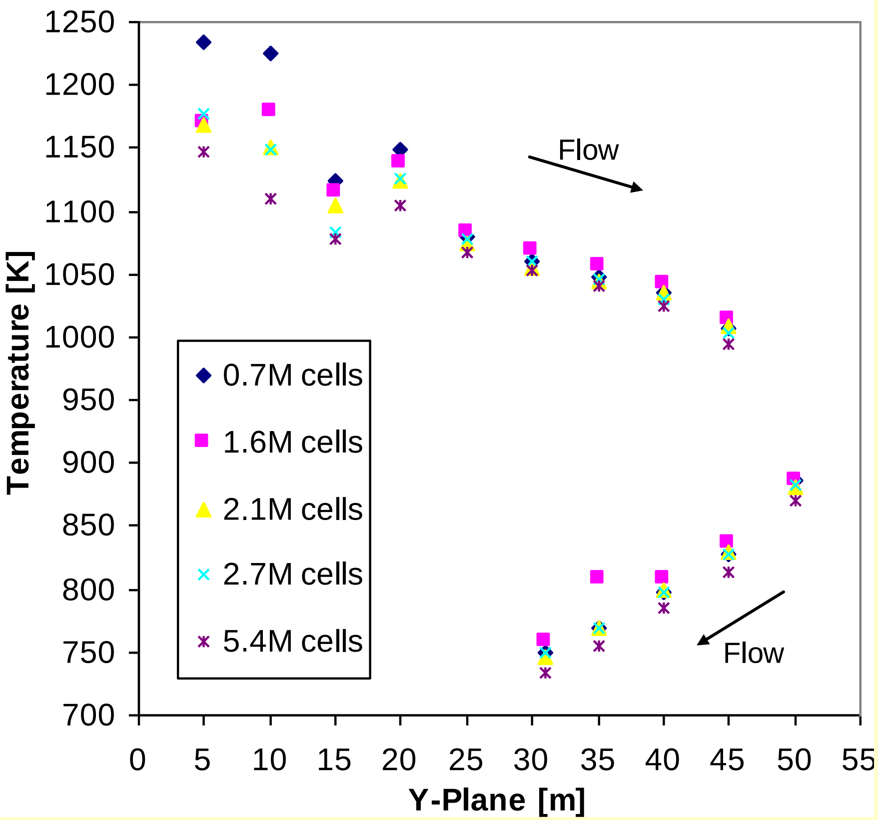
Dashed blue and green lines show the temperature error bars for the 5.4M and 1.6M cells, respectively

Error bars obtained from temperature monitor at central point in burners zone

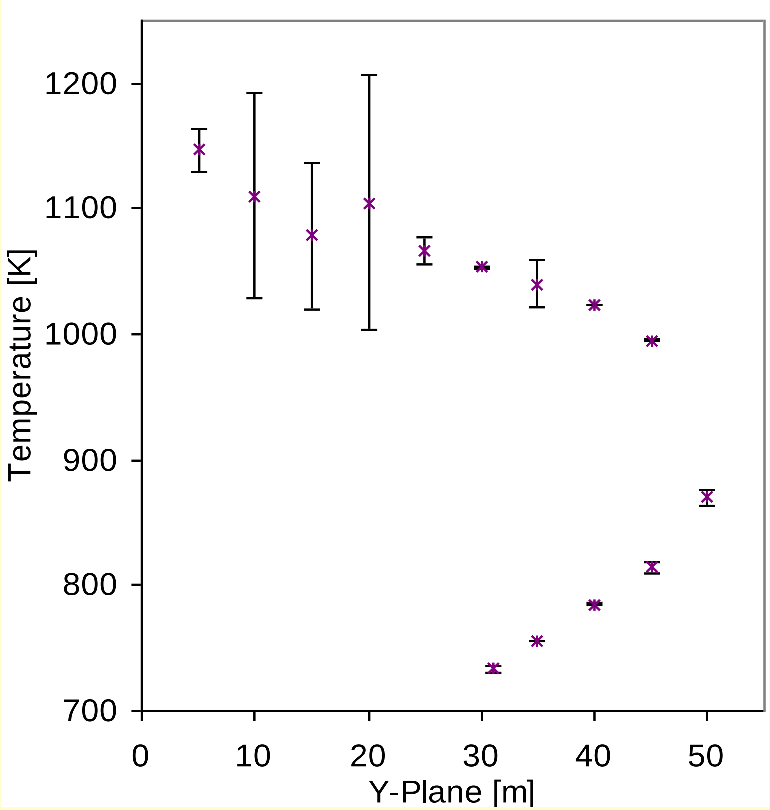


Temperature monitor at a burner point in the 5.4M cells-grid

Discretization Uncertainty

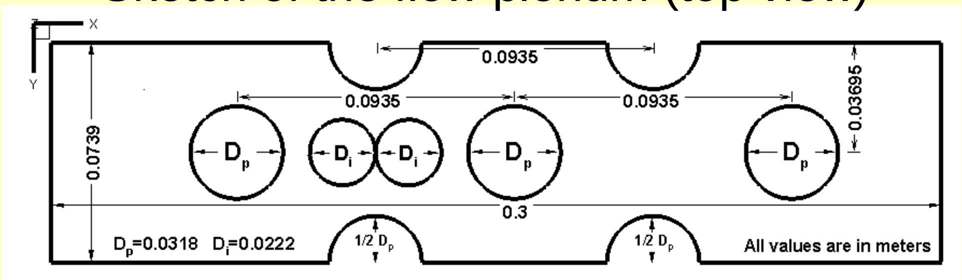


Bulk-Mean Temperatures at Horizontal Cross Planes

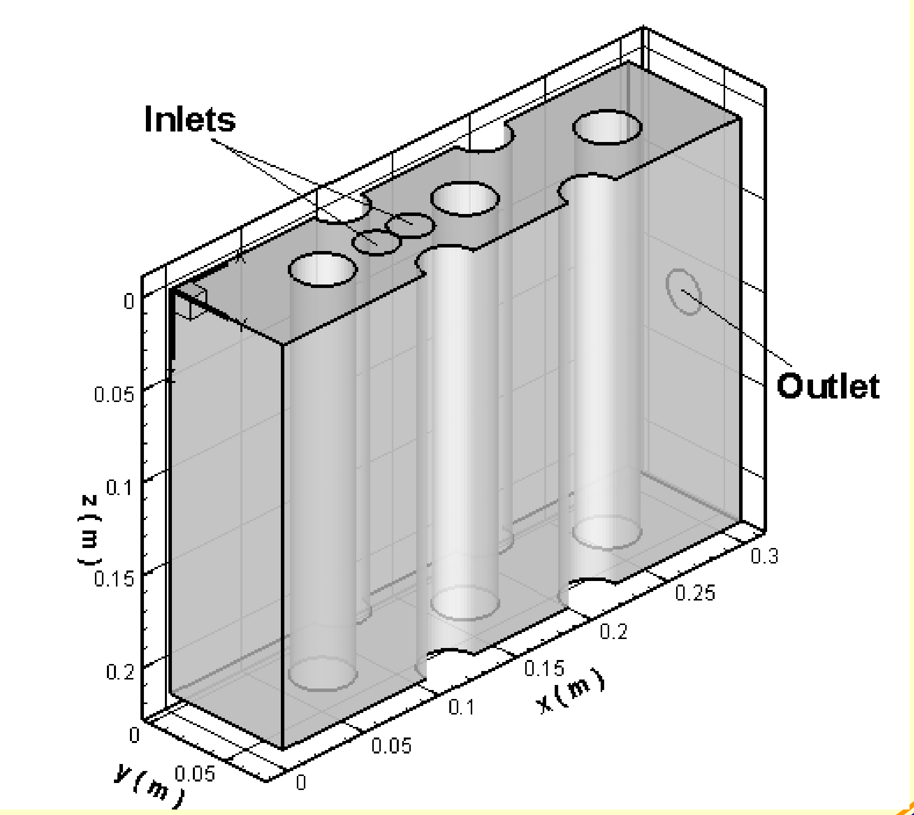


Local apparent order 0.6-14.6, p_{ave}=5.46
Maximum discretization uncertainty 9.3% (±102K)

Sketch of the flow plenum (top view)



3-Dimensional perspective



3D structured boundary layers, extending 20% of the pole diameter, Dp

Average $y^+ < 5$

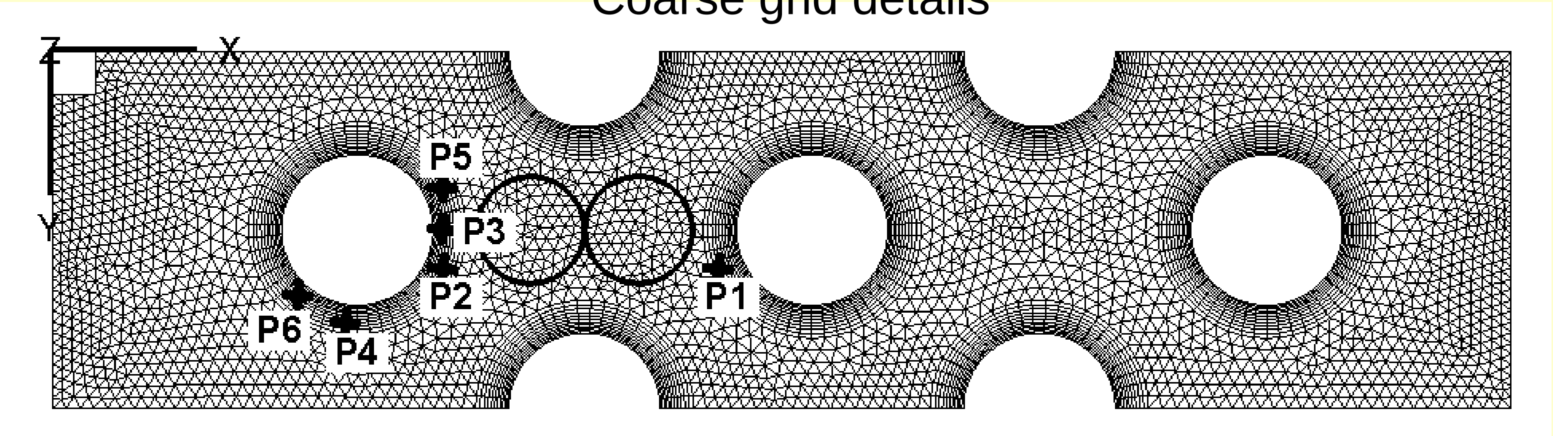
Enhanced wall treatment in the near wall region

Rest of the domain : unstructured tet-mesh

Mesh coarsening factor, *r*, of ~1.5

593,928 (coarse), 873,493 (medium), and 1,186,944 (fine) cells

Coarse grid details



Flow Details

	Low-Re	High-Re
Inlet velocity of a single jet	1.611×10^{-1} m/s	5.660×10^{-1} m/s
Total flow rate	1.246×10^{-4} m^3/s	4.382×10^{-4} m^3/s
Average plenum velocity	7.739×10^{-3} m/s	2.720×10^{-2} m/s
Flow through time (FTT)	32s	9s
Total execution time	450s (~14 FTT)	250s (~28 FTT)
Pole Reynolds number	245	431

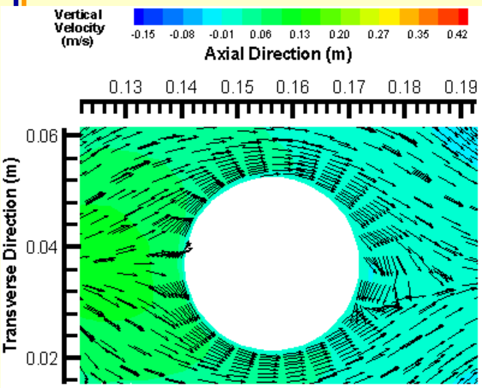
Turbulence models : RNG k-ε, SST k-ω

Residuals : 1×10^{-4}

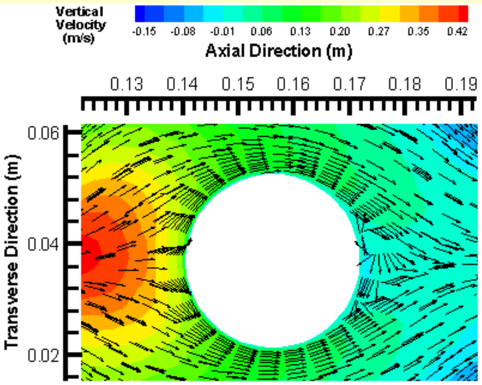
Time step : 1×10^{-3}s

Numerical Schemes : 2nd order upwind (Conv.), 2nd order central (Diff.)
2nd order (Pressure) , 1st order (time)

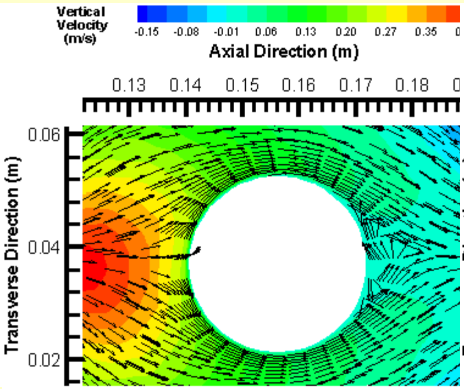
Separation from middle rod



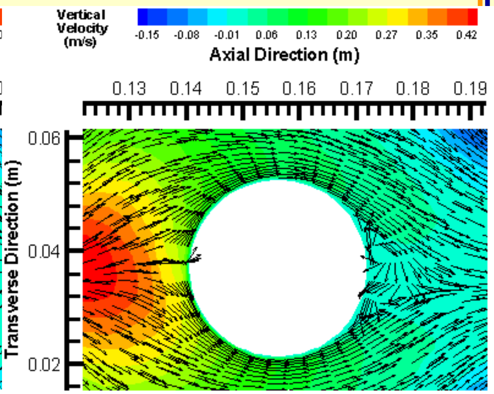
SST-C-L



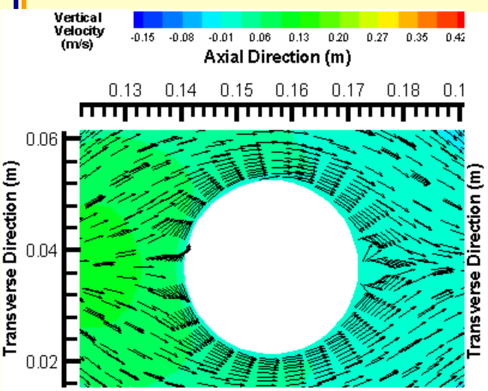
SST-C-H



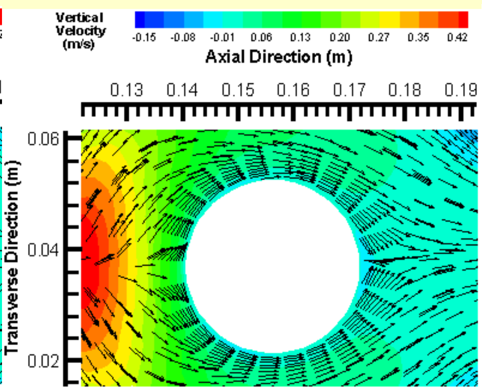
SST-M-H



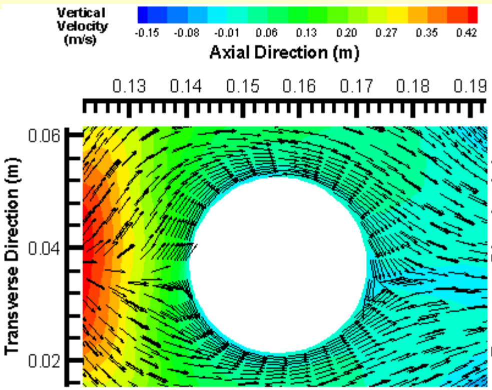
SST-F-H



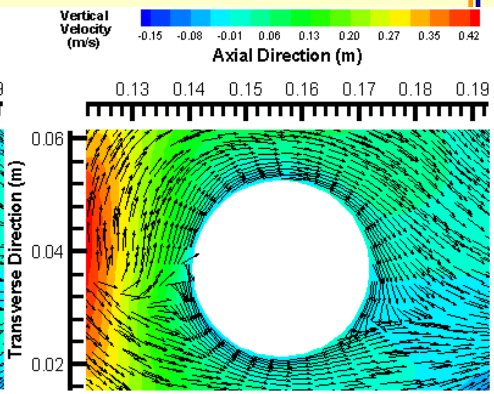
RNG-C-L



RNG-C-H

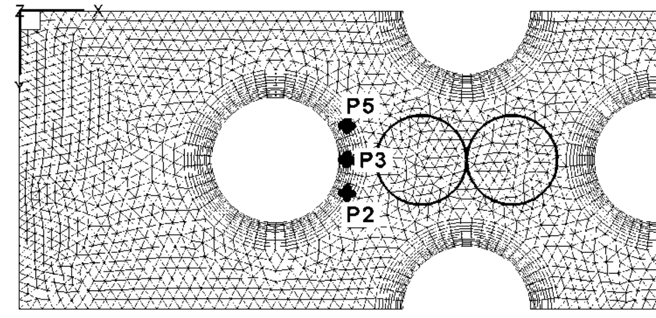


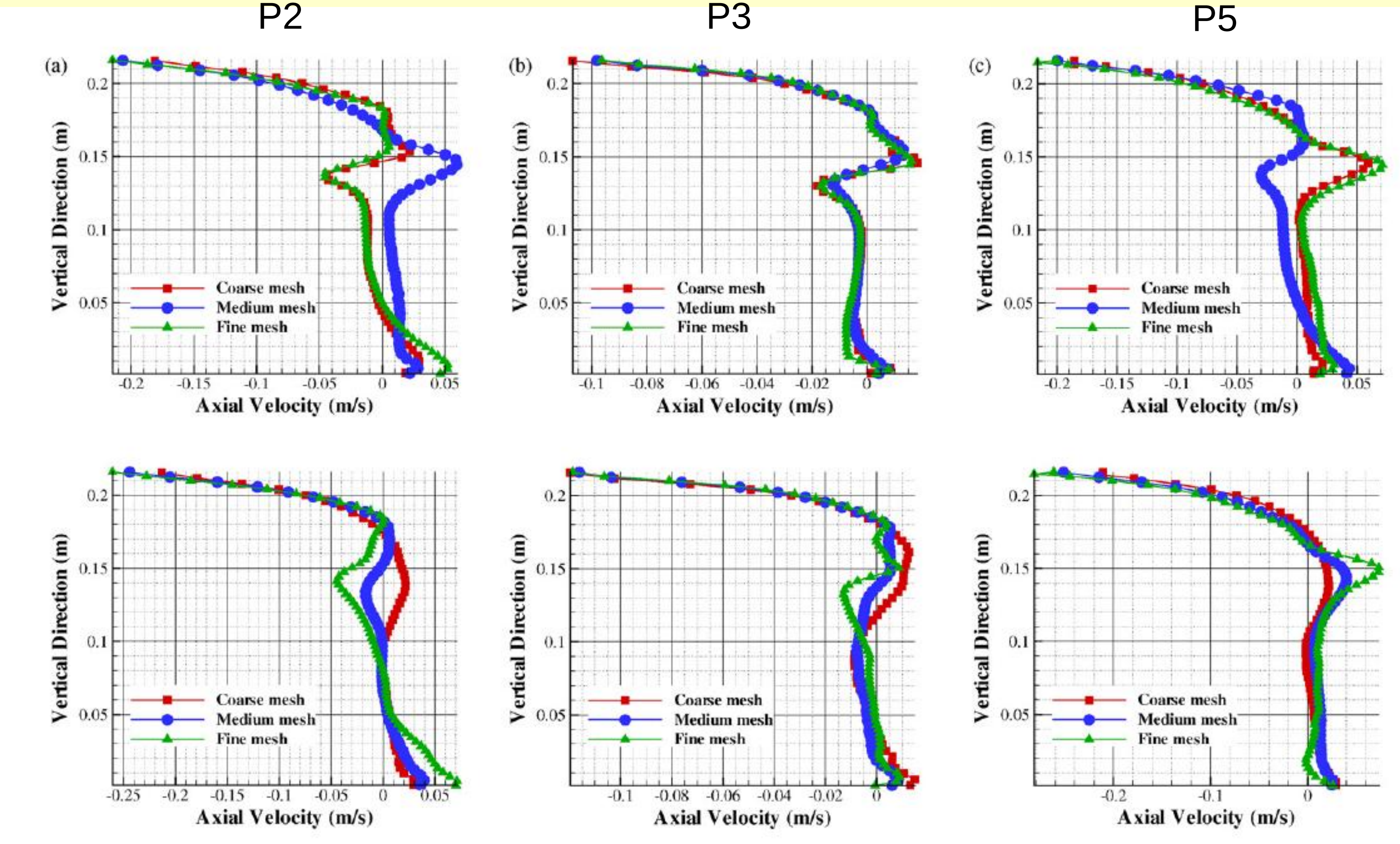
RNG-M-H



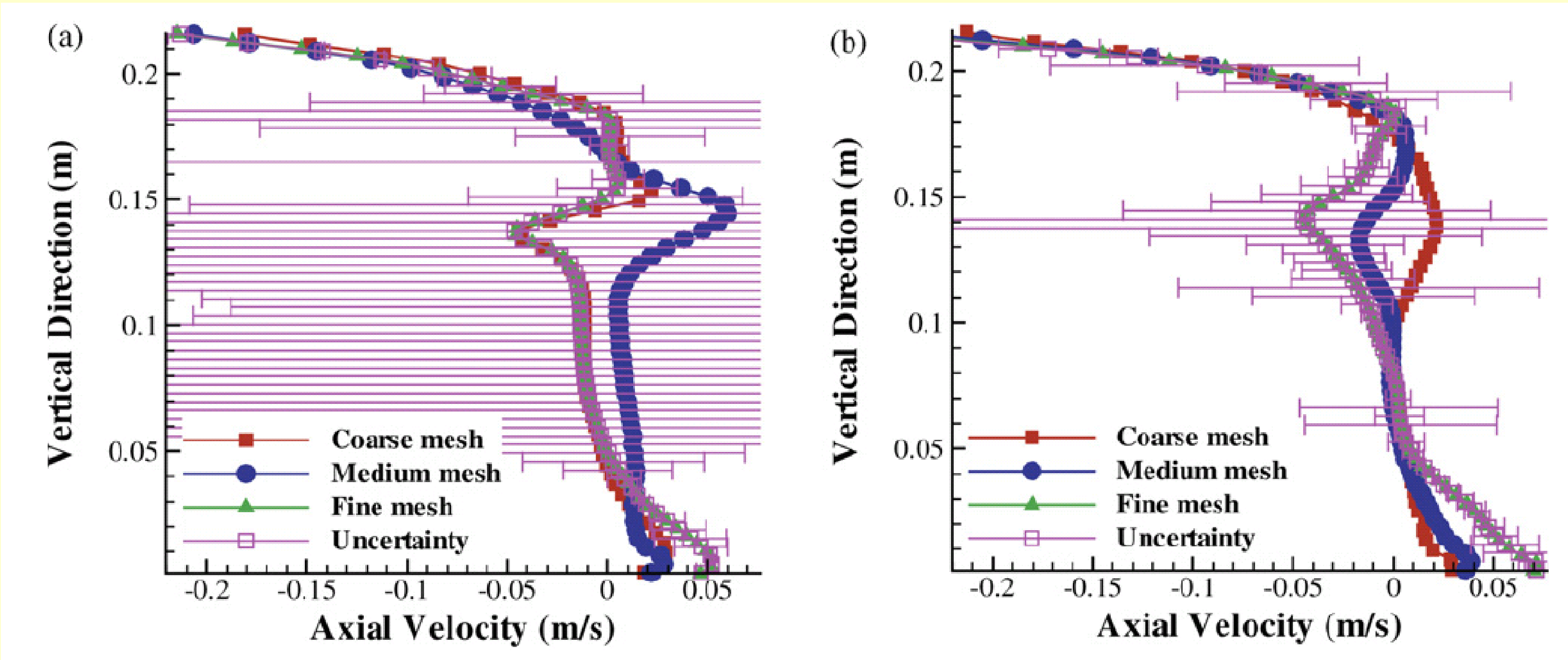
RNG-F-H

Ensemble ave. of axial velocity profiles over the last 100s SST k-ω (top row), and RNG k-ε(bottom row)





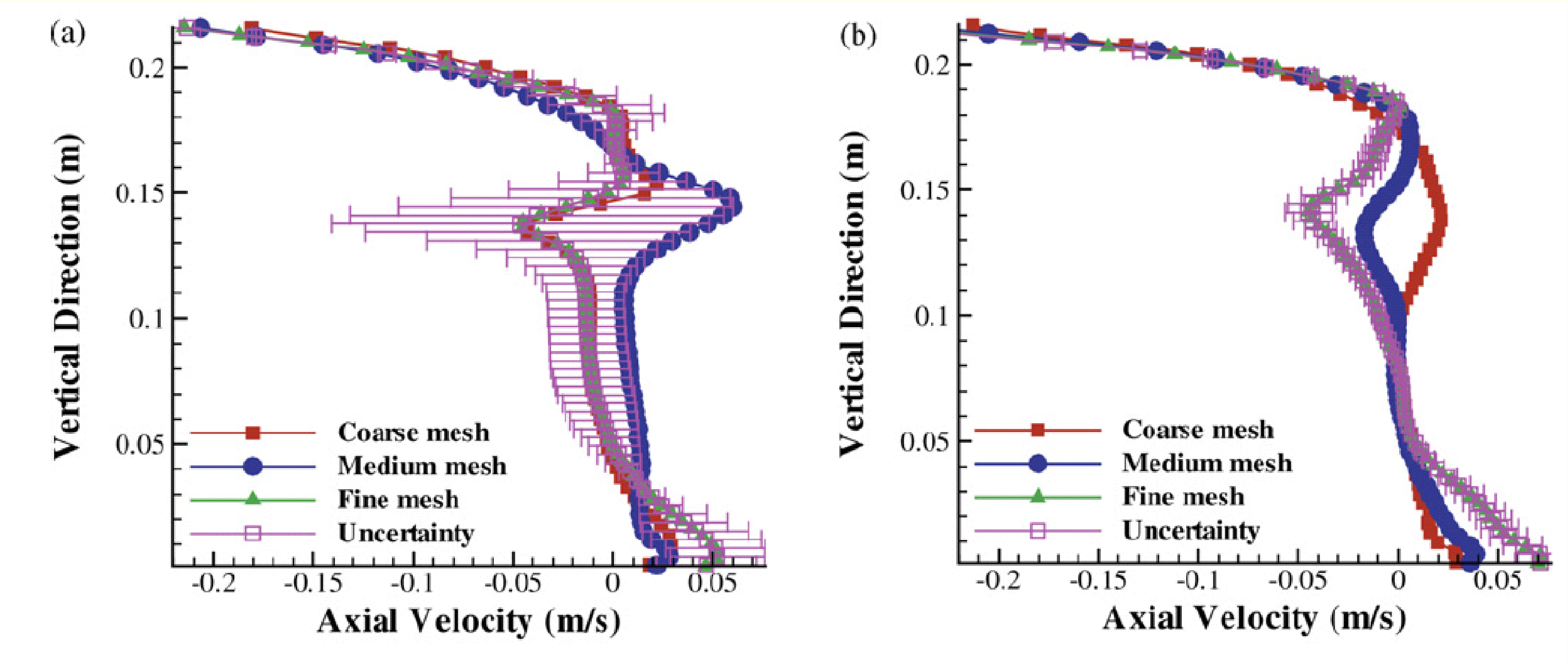
Predicted uncertainties with JFE method at P2



SST k-ω model

RNG k-ε *model*

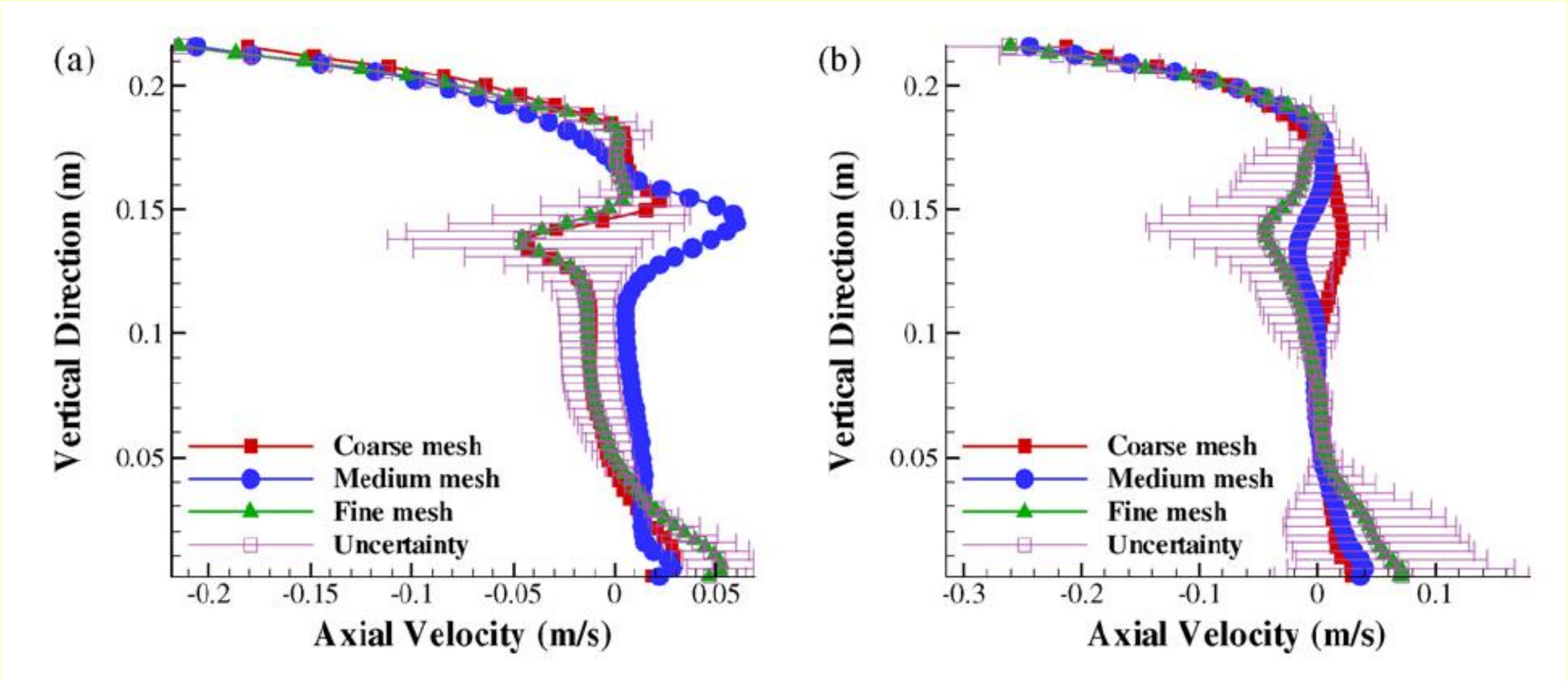
Predicted uncertainties with JFE method and averaged *p* (order) at P2



SST k-ω model

RNG k-ε *model*

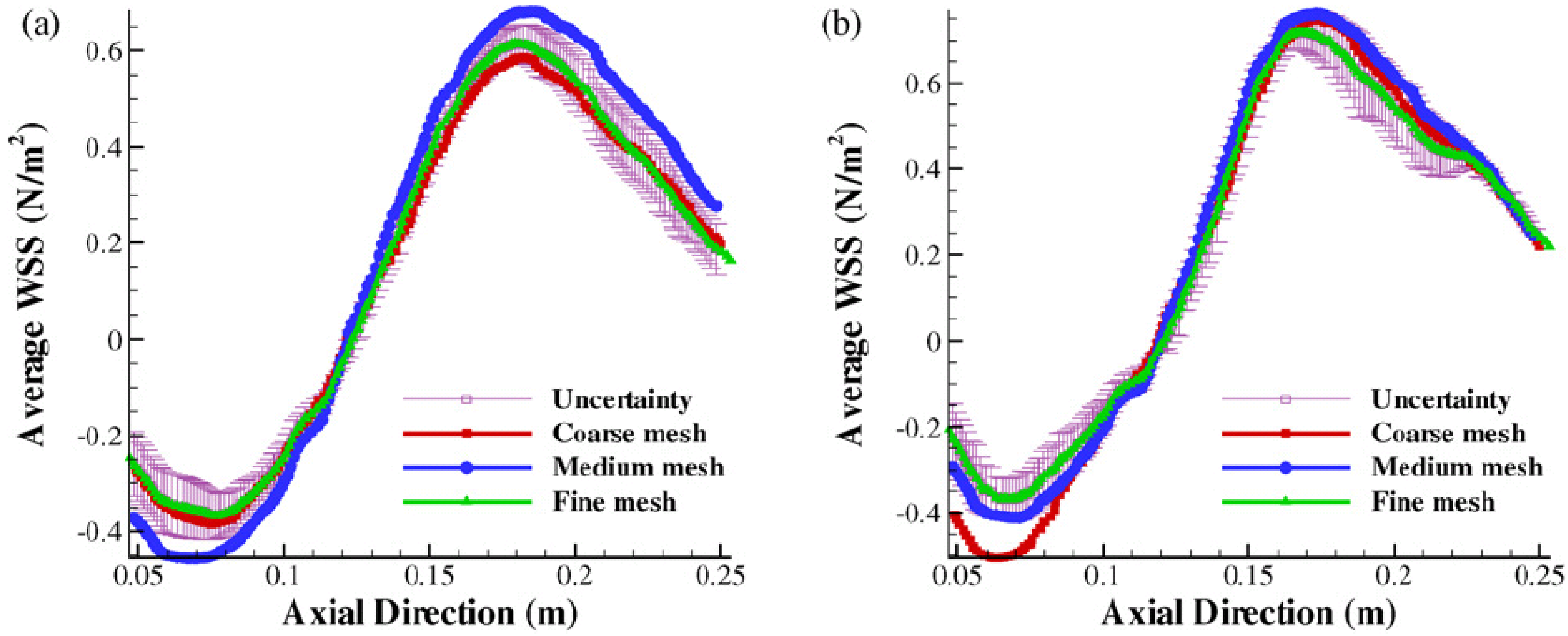
Predicted uncertainties with AES method at P2



SST k-ω model

RNG k-ε *model*

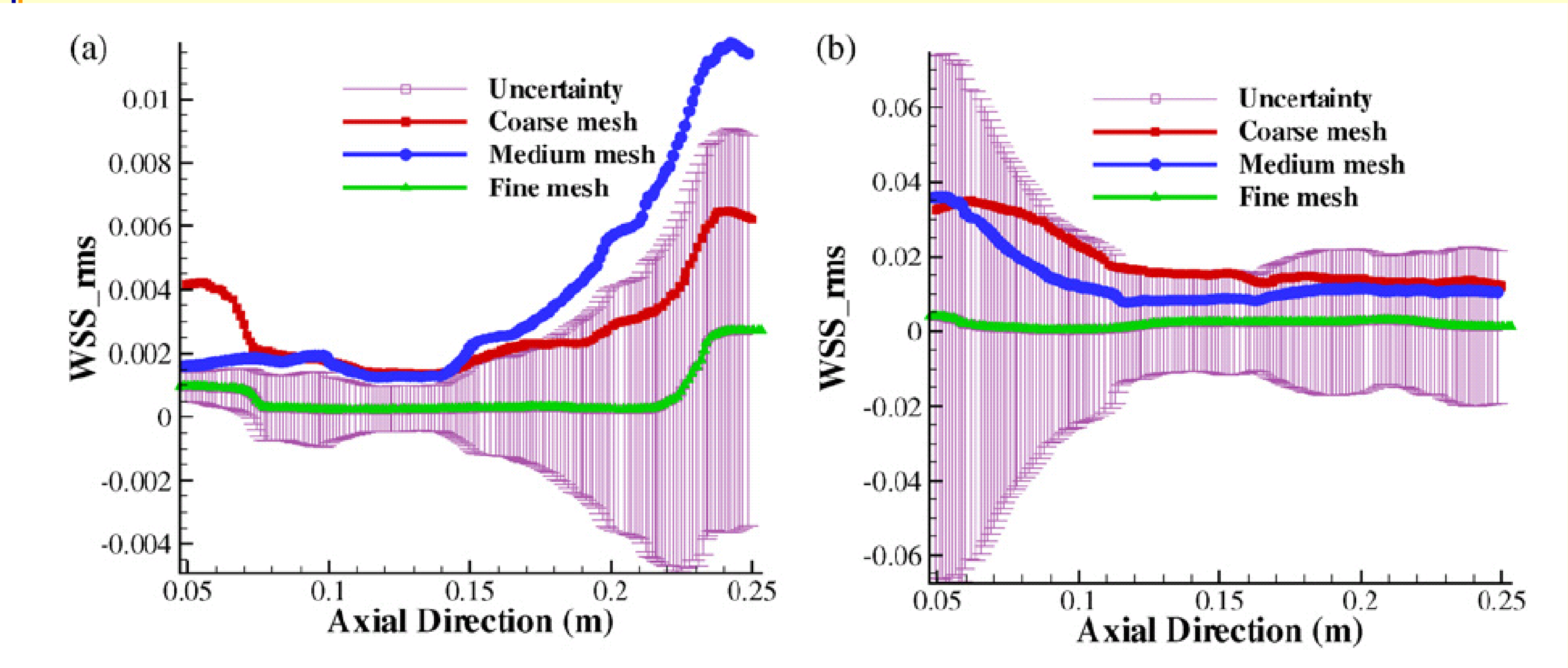
Spatially filtered wall shear stress profiles and uncertainties by AES method at y = 0.018475m z = 0.2175 m



SST k-ω model

RNG k-ε *model*

Spatially filtered *rms* wall shear stress profiles calculated using ensemble averaging along with the uncertainties by AES method at y = 0.018475m z = 0.2175 m



SST k-ω model

RNG k-ε *model*

ETE vs. RE

Ref: Celik & Hu, 2004

- Richardson extrapolation (RE)
  - Popular, relatively reliable (+)
  - At least three sets of grid, expensive (-)
  - Difficult to identify asymptotic range (-)
  - Does not work for oscillatory grid convergence (-)

- Error transport method (ETE)
  - No extra effort in grid generation (+)
  - Can be solved using the same scheme (+)
  - Can be used as a post-processing tool for steady problems(+)
  - Additional recourses for code development (-)
  - Difficulty in determining source term of ETE (-)
  - Reliability still under investigation (-)

Literature review of ETE

- Roache (1993 & 1998)
- Van Straalen et al. (1995)
- Zhang et al. (1997)
- Wilson & Stern (2001)
- Celik & Hu (2002, 2003)
- Qin & Shih (2003)

Error Transport Equation (ETE)

Non-linear: $L(\phi) = 0$

Linearized: $L_h(\tilde{\phi}) = 0$ (1)

$L_h(\phi) = R = \tau(\phi)$ (2)

L : differential operator (PDE)

L_h : difference operator (FDE)

ϕ : exact solution to PDE

$\phi^{\sim}$: numerical solution

R : residual

error is defined as: $\varepsilon = \phi - \tilde{\phi}$

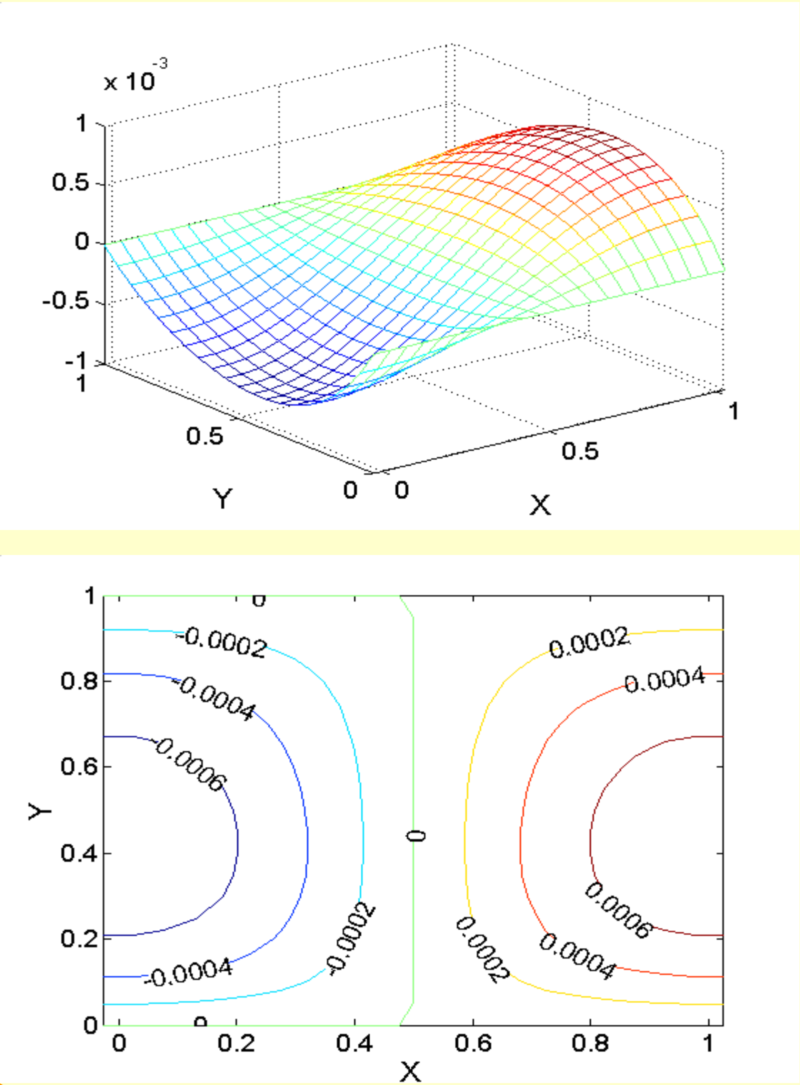
ETE: $L_h(\varepsilon) \equiv L_h(\phi) - L_h(\tilde{\phi}) = \tau(\phi)$

τ represents the truncation error of a discretized equation, i.e. the *error source term*

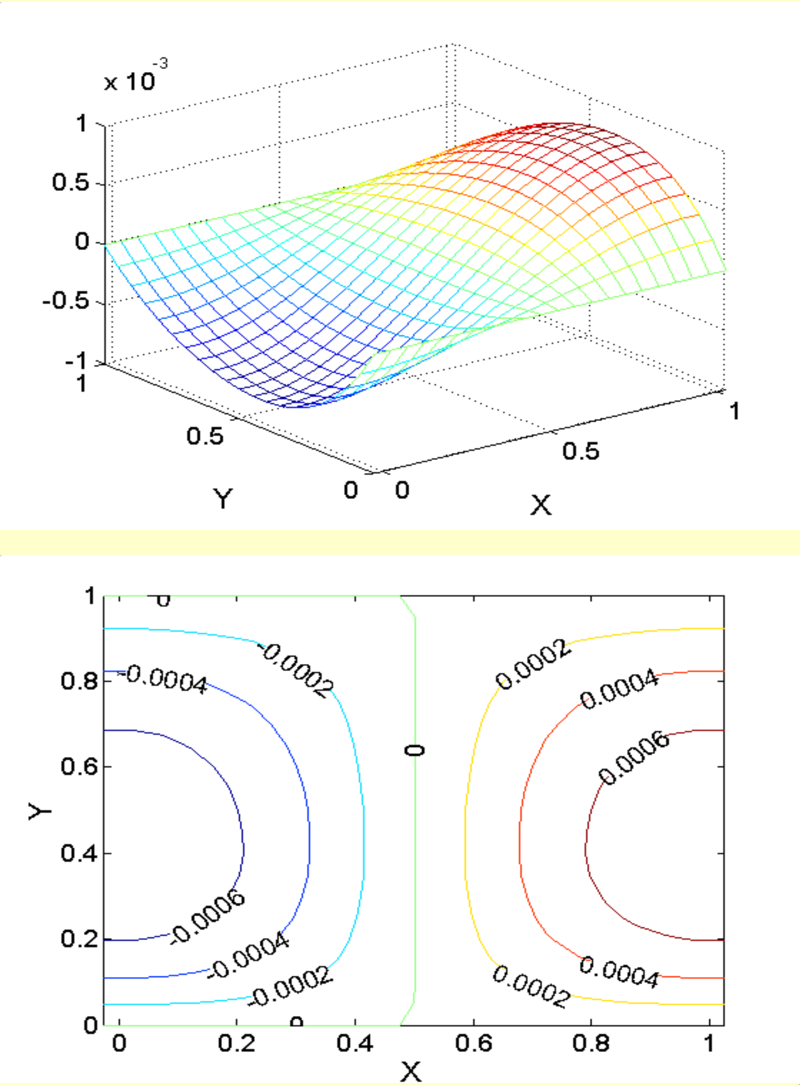
2D Poison Equation:

Central difference Scheme

Exact error

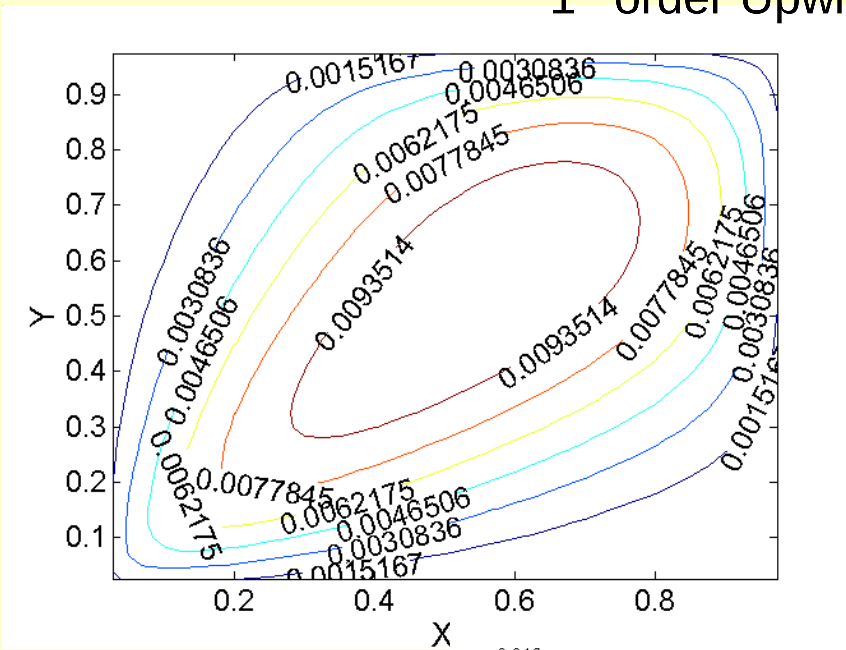


ETE error

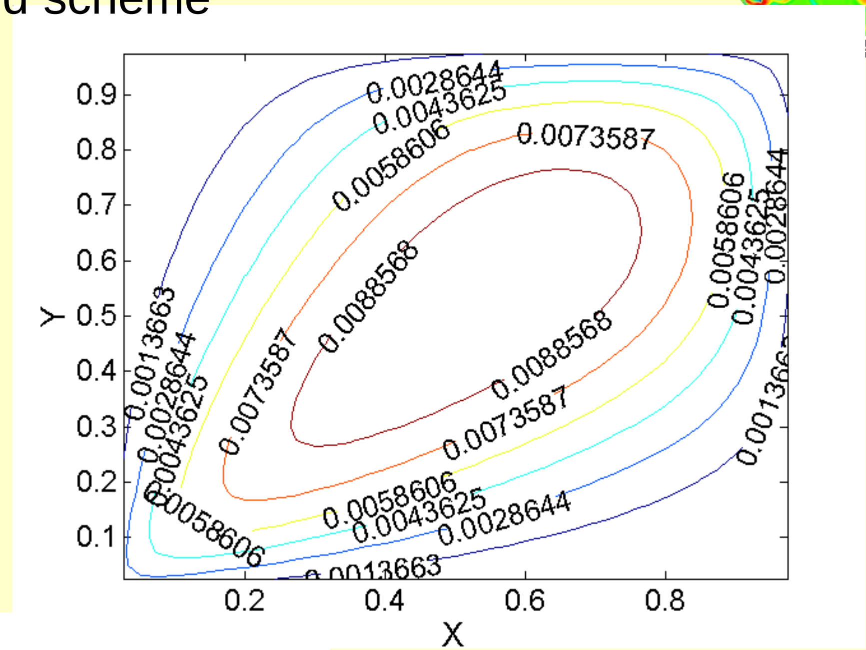


2D Steady Convection Diffusion

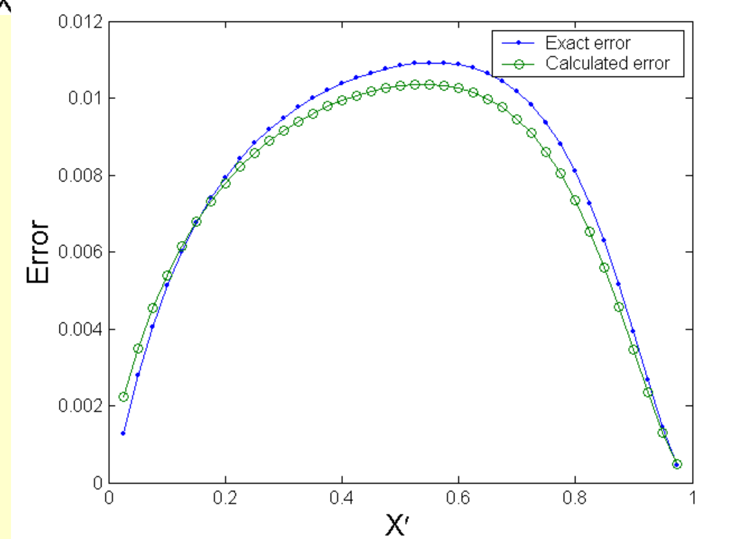
1st order Upwind scheme



Exact error



Calculated error

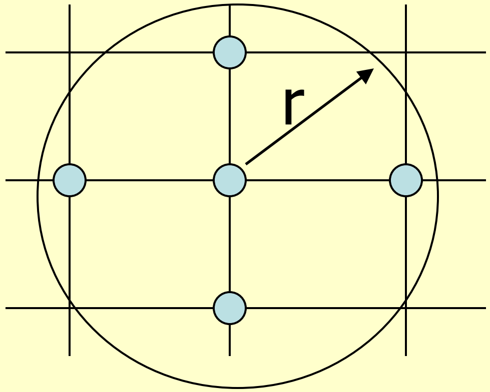


Line plot along diagonal

Generalized Derivation of Error Source

$$\begin{pmatrix} \text{implicit} \\ \text{coefficient} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \phi^{new} \end{pmatrix} = \begin{pmatrix} \text{explicit} \\ \text{coefficient} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \phi^{old} \end{pmatrix}$$

Influence circle →



Need to know:

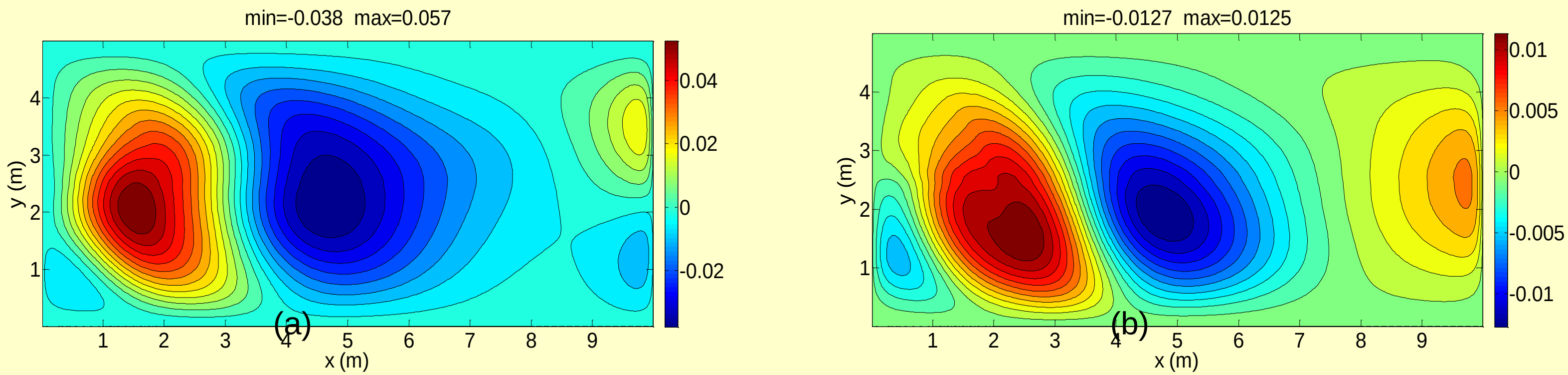
1. Access to the coefficient matrix
2. Influence circle (or radius)

Conclusions

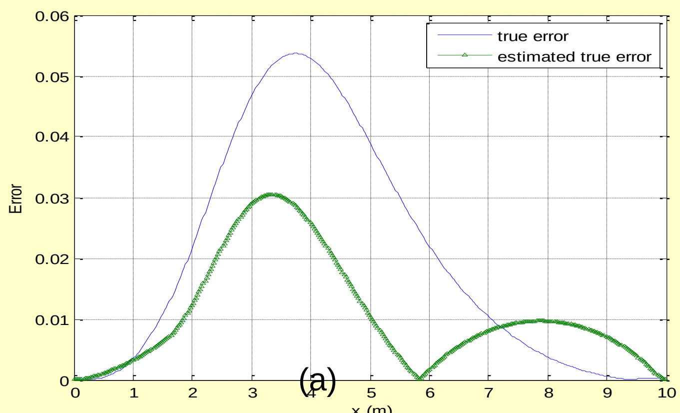
- For RANS, methods based on Richardson extrapolation are preferred for their robustness, however they do not always work and it is not easy to detect where an when they will fail.
- Although there is evidence that time step can be simply treated as another discretization parameter, it is much safer to use a relatively small time step so as to minimize its effect compared to space discretization.
- To quantify discretization errors at least 3-4 grid calculations are needed (may be 4-5 sets for oscillatory convergence); It may be erroneous to assume monotonic convergence just by observing the behavior of three or four points.

- ***A hybrid of extrapolation and ETE methods is the way to go!***

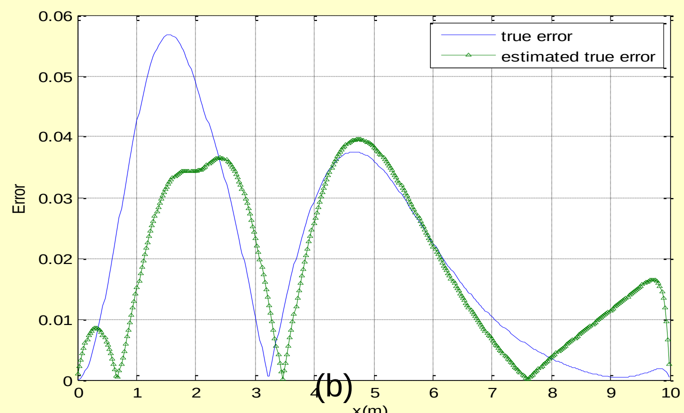
Example: Hybrid AES & ETE



Error in axial velocity (a) True error on medium grid, (b) Approximate error between fine and medium grid



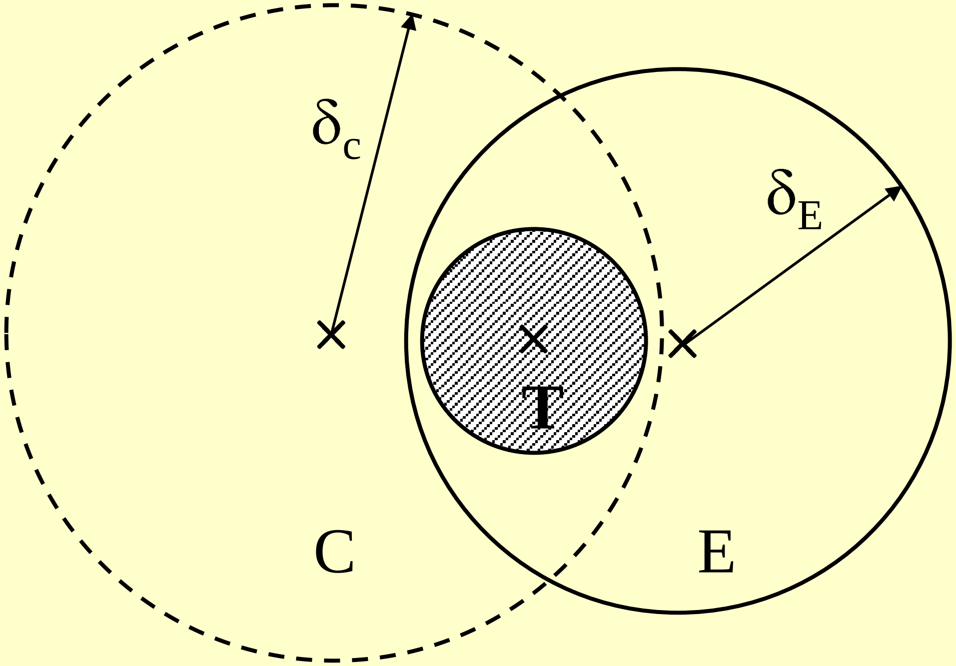
Estimated true error in axial velocity at (a) x= 3.5m, (b) y=1m, (c) y=3.5m



Estimated true error in vertical velocity at (a) x= 1.5m, (b) x=5m, (c) y=2m

Challenge

Predict the 'truth' within an acceptable confidence interval without knowing the 'truth'



"What can not be computed is meaningless!"

(Davies, 1992)

T = 'truth' ± fuzziness about truth

References

1. Anderson, W.K., and Venkatakrishnan, V., "Aerodynamic design optimization on unstructured grids with continuous adjoint formulation," AIAA paper 97-0643, 1997.
2. Becker, R. and Rannacher, R., "Weighted a posteriori error control in finite element methods," Technical report, Universitat Heidelberg, preprint No.96-1, 1994.
3. Celik I, Karatekin O., "Numerical Experiments on Application of Richardson Extrapolation with Nonuniform Grids," ASME Journal of Fluids Engineering, 1997; 119:584-590.
4. Celik, I., Cehreli, Z.N. and Yavuz, I., "Index of Resolution Quality for Large Eddy Simulations," 2003 ASME FED Summer Meeting, July 6-10, Honululu, Hawai, USA, 2003.
5. Celik, I. and Hu, G., "Single Grid Error Estimation Using Error Transport Equation", ASME Journal of Fluids Engineering, 2004; 126:778-790.
6. Celik, I., Li, J., Hu, G., and Shaffer, C., "Limitation of Richardson Extrapolation and Some Possible Remedies," Journal of Fluids Engineering, July, 2005.
7. Celik, I., Cehreli, Z.N. and Yavuz, I., "Index of Resolution Quality for Large Eddy Simulations," accepted for publication in Journal of Fluids Engineering, 2005.
8. Celik, I., and Li, J., "Assessment of Numerical Uncertainty for the Calculations of Turbulent Flow over a Backward Facing Step" Submitted to Int. J. Num. Methods in Fluids, accepted, 2005.
9. Cervantes, M.J. and Engstrom, F.T., "Factorial Design Applied to CFD," Journal of Fluids Engineering, 2004; 126:791-798.
10. Cehreli Z., Ph. D Dissertation, West Virginia University, Morgantown, West Virginia, 2003.
11. Eça L, Hoekstra M. eds., "Proc. of Workshop on CFD Uncertainty Analysis," Lisbon, 21-22 October, 2004.
12. ERCOFTAC Classic Collection Database – *http://cfd.me.umist.ac.uk/ercoftac*, accessed on December 11, 2005.
13. Ferziger, J.H., 1989 "Estimation and Reduction of Numerical Error," ASME Winter Annual Meeting, San Fransisco, December, 1989.

References

14. Ghanem, R., Organizer, Workshop on Elements of Predictability, The Johns Hopkins University, November, 13-14, 2003.
15. Giles, M.B. (1997) "On Adjoint Equations for error analysis and optimal grid adaptation in CFD," Oxford Univ. Computing Laboratory, http://www.comlab.ox.ac.uk
16. Geurts, B.J., Froehlich, J., "A framework for predicting accuracy limitations in large eddy simulation", Physics of Fluids, 2002; 14:41-44.
17. Gullbrand, J., and Chow, F.K., "Investigation of numerical errors, subfilter-scale models, and subgrid-scale models in turbulent channel flow simulations," Center for Turbulence Research Proceedings of the Summer Program, Stanford University, 2002.
18. Hanson K.M. and Hemez, F.M., Editors, http://library.lanl.gov/ Proceedings of the 4th International Conference on Sensitivity Analysis of Model Output (SAMO), Sante Fe, NM, March 8-11, 2004.
19. Hay, A., and Visonneau, M., "CFD uncertainty Analysis for Turbulent Flows using Error Equation Method," Proc. of Workshop on CFD Uncertainty Analysis, Lisbon, 21-22 October, 2004.
20. Hu, G., Ph. D Dissertation, West Virginia University, Morgantown, West Virginia, 2005.
21. Huebsch, W.W., "Two-Dimensional Simulation of Dynamic Surface Roughness for Aerodynamic Flow Control," accepted, AIAA Journal of Aircraft, May, 2005.
22. Klein, K., "An attempt to assess the quality of large eddy simulations in the context of implicit filtering" 2nd International Workshop on Trends in Numerical and Physical Modeling for Turbulent Processes in Gas Turbine Combustors, Darmstadt, 2004.
23. Meyers. J., Geurts, B.J., and Baelmans, M., "Database Analysis of Errors in Large Eddy Simulation", Physics of Fluids, Vol. 15 No. 9, 2003.1
24. Li, J., "Critical issues for predicting worker exposure in a wind tunnel," Ph.D. Thesis, 2005.
25. Rebba, R. Mahadevan, S., and Huang, S., "Validation and Error Estimation of Computational Models," http://library.lanl.gov/ Proceedings of the 4th International Conference on Sensitivity Analysis of Model Output (SAMO), Sante Fe, NM, March 8-11, 2004.

References

26. Orozco, C., Claramunt, K., Consul, R., and Oliva, A., "Finite Volume Computation and Verification of Fluid Flow and Heat Transfer in Domains with Moving Boundaries," European Congress on Comp. Meth. Appl. Sci. Eng. , Jyvaskyla, Spain, July 24–28, 2004.
27. Roache, P.J., "Verification and Validation in Computational Science and Engineering" Hermosa Publishers, Albuquerque, 1998.
28. Roache, P.J., "Perspective: a Method for Uniform Reporting of Grid Refinement Studies," J.of Fluids Engineering, 1994; 116:405-413.
29. Shaffer, C., Master thesis, West Virginia University, Morgantown, West Virginia, 2005.
30. Sonar, T., "Strong and Weak Norm Refinement Indicators Based on the Finite Element Residual for compressible flow computations," Impact of Comp. Sci. Eng., 1993; 5: 11-127.
31. Soria, M., Trias, F.X., Perez-Segarra, C.D., and Oliva, A., "Direct Numerical Simulation of a Three Dimensional Natural-Convection Flow in a Differentially Heated Cavity of Aspect Ratio 4", Numerical Heat Transfer, Part A, 45: 649-673, 2004.
32. Süli, E., "A posteriori error analysis and global error control for adaptive finite element approximations of hyperbolic problem," Proc. of the 16th Biennial Conf. in Num. Anal., 1996.
33. Tafti, D., Private communication, High Performance Computational Fluid Thermal Science & Engineering Lad, Mechanical Engineering Department, Virginia Tech, 2005.
34. Tomlin, A.S., "The Evaluation of Combustion Mechanisms using Local and Global Sensitivity and Uncertainty Methods," http://library.lanl.gov/ Proceedings of the 4th International Conference on Sensitivity Analysis of Model Output (SAMO), Sante Fe, NM, March 8-11, 2004.
35. Viswanathan A.K., and Tafti, D.K., "Detached eddy simulation of turbulent flow and heat transfer in a ribbed duct," Proc. of ASME HT/FED Summer Conf., July 11-15, Charlotte, 2004.
36. Zhang, Z., Pelletier, D., Trapanier, J.Y., and Camarero, J.Y., "Numerical assessment of error estimates for Euler equations," AIAA Journal, 2001; 39:1706-1715

References

37. I. Celik, M. Klein, M. Freitag and J. Janicka, "Assessment Measures for URANS/DESLES: An overview with applications," Journal of Turbulence, in Press.

38. I. Celik, "Assessment Measures for LES Applications," European Fluids Engineering Meeting, July 17-20, 2006, Miami, Florida.

39. Francisco Elizalde, I. Celik, Suryanarayana Pakalapati, "Uncertainty Estimation for Numerical Wall Turbulent Flows," European Fluids Engineering Meeting, July 17-20, 2006, Miami, Florida.

Additional References

•Cadafalch, J., Perez-Segarra, C.D., Consul, R. and Oliva, A. (2002) “Verification of Finite Volume Computations on Steady-State Fluid Flow and Heat Transfer,” ASME Journal of Fluids Engineering, Vol. 124, pp.11-21.

•Celik, I., Chen, C.J., Roache, P.J. and Scheurer, G. Editors. (1993), “Quantification of Uncertainty in Computational Fluid Dynamics,” ASME Publ. No. FED-Vol. 158, ASME Fluids Engineering Division Summer Meeting, Washington, DC, 20-24 June.

•Celik, I. and Zhang, W-M (1993) “Applicaton of Richardson Extrapolation to Some Simple Turbulent Flow Calculations,” Proceedings of the Symposium on “Quantification of Uncertainty in Computational Fluid Dynamics,” Editors: I. Celik et al. ASME Fluids Engineering Division Spring Meeting, Washington, C.C., June 20-24, pp. 29-38.

•Celik, I. and Karatekin, O. (1997), “Numerical Experiments on Application of Richardson Extrapolation with Nonuniform Grids, ” ASME Journal of Fluids Engineering, Vol. 119, pp.584-590.

•Celik, I., Li, J., Hu, G., and Shaffer, C. (2004) “Limitations of Richardson Extrapolation and Possible Remedies for Estimation of Discretization Error” Proceedings of HT-FED2004, 2004 ASME Heat Transfer/Fluids Engineering Summer Conference, July 11-15, 2004, Charlotte, North Carolina USA, HT-FED2004-56035

•Eca, L. and Hoekstra, M. (2002), “An Evaluation of Verification Procedures for CFD Applications,” 24th Symposium on Naval Hydrodynamics, Fukuoka, Japan, 8-13 July.

•Eca, L. and Hoekstra, M. (2003), “Uncertainty Estimation :A Grand Challenge for Numerical Ship Hydrodynamics” 6th Numerical Towing Tank Symposium, Rome, September 2003

•Eca, L. et al., (2004) “Workshop on Numerical Uncertainty Estimation” Lisbon, Portagul 2004

•FLUENT company 2004 FLUENT 6.0 User Reference Manual

•Phillips, G.M. and Taylor, P.J. (1973) “Theory and Applications of Numerical Analysis” Academic Press: London and New York, pp.345

Additional References

•Richardson, L.F. (1910) “The Approximate Arithmetical Solution by Finite Differences of Physical Problems Involving Differential Equations, with an Application to the Stresses In a Masonary Dam,” Transactions of the Royal Society of London, Ser. A, Vol. 210, pp. 307-357.

•Richardson L.F. and Gaunt, J. A. (1927) “The Deferred Approach to the Limit,” Philos. Trans. R. Soc. London Ser. A, Vol. 226, pp. 299-361.

•Roache, P.J.(1998) “Verification and Validation in Computational Science and Engineering,” Hermosa Publishers, Albuquerque

•Stern, F., Wilson, R. V., Coleman, H. W., and Paterson, E. G. (2001), "Comprehensive Approach to Verification and Validation of CFD Simulations - Part 1: Methodology and Procedures," ASME Journal of Fluids Engineering, Vol. 123, pp. 793-802, December.

•C. I. Roy (2010) “Review of Discretization Error Estimators in Scientific Computing, AIAA Paper, 092407

•William L. Oberkampf and Christopher J. Rey, (2010) “Verification and Validation in Scientific Computing”, Cambridge